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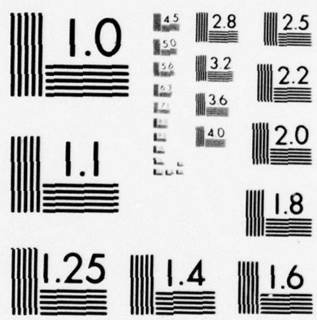
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DYNAMIC THEORY OF PRODUCTION CORRESPONDENCES: PART III

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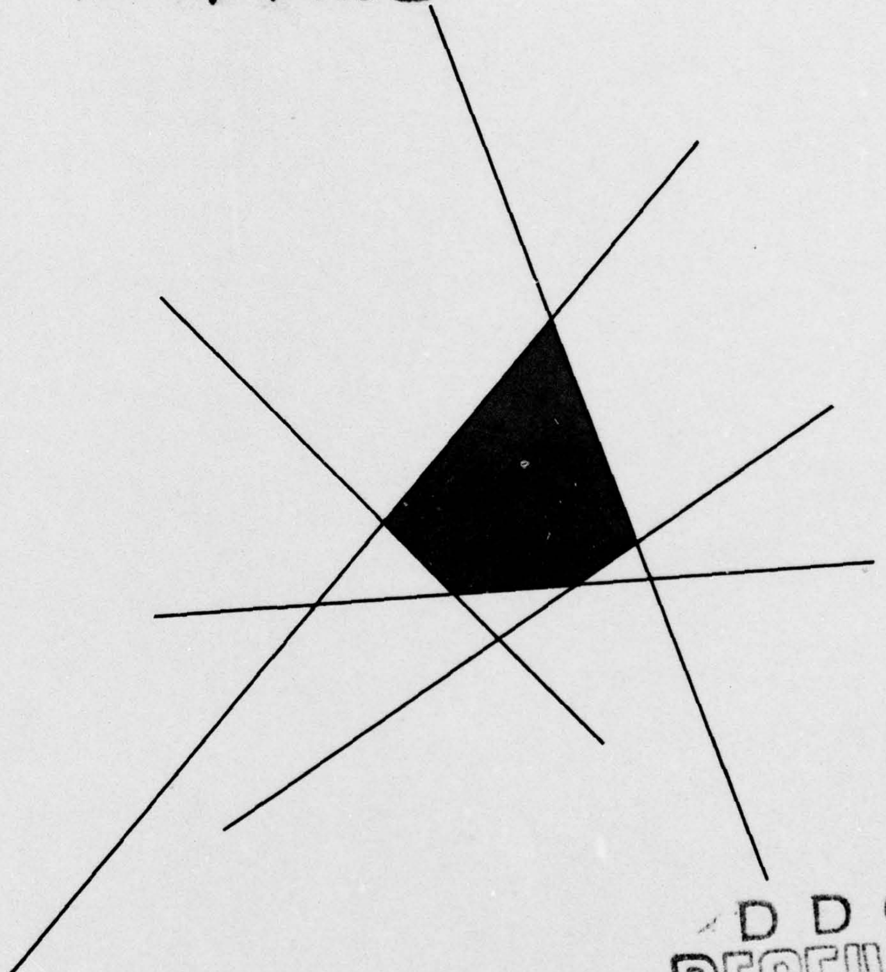
by
RONALD W. SHEPHARD
and
ROLF FÄRE



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DYNAMIC THEORY OF PRODUCTION CORRESPONDENCES†

PART III

by

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ABSTRACT

Chapters 7 thru 9 of a monograph on a Dynamic Theory of Production Correspondences are presented. Dualities and Shadow Pricing, Index Functions for Production Theory and Indirect Dynamic Production Correspondences are discussed.

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CHAPTER 7

DUALITIES AND SHADOW PRICING

The dualities to be considered here are between: (a) Minimal Cost Functional and Distance Functional for Input Structures, (b) Maximal Revenue Functional and Distance Functional for Output Structure. Globally strong forms of these dualities take the forms: (a) Minimal Cost and Distance Functional are expressable in terms of each other by dual minimal problems, (b) Maximal Revenue and Distance Functional are expressable in terms of each other by dual maximal problems. These strong forms require the addition of rather strong additional assumptions beyond the weak axioms for the related dynamic production correspondences. Globally weak forms involving inequalities for the equality expressing the distance functionals apply under the weak axioms. Partial dualities in the strong form are possible under the weak axioms.

The shadow pricing is formed by use of the dual functionals in the weak form. Such prices are a dynamic generalization of the usual shadow prices of static mathematical programming.

7.1 Duality Between Minimal Cost Functional and Distance Functional for Input Correspondence

For the analysis to follow it is convenient to introduce for $u \in (L_\infty)^m_+$, $x \in (L_\infty)^n_+$, the functional

$$(7.1\text{-}1) \qquad \bar{\underline{\Psi}}^*(u,x) := \operatorname*{Inf}_p \left\{ \langle p,x \rangle : \mathbb{K}(u,p) \geqq 1 \, , \, p \in (L_1)^n_+ \right\} ,$$

and define a subset $\mathbb{L}^*(u)$ for $u \in (L_\infty)^m_+$ by

(7.1-2) $$\mathbb{L}^*(u) := \left\{x \in (L_\infty)^n_+ : \underline{\bar{\Psi}}^*(u,x) \geqq 1\right\} .$$

The following proposition holds:

Proposition (7.1-1): (Shephard, 1970:a) $\mathbb{L}(u) \subset \mathbb{L}^*(u)$

If $u = 0$, $\mathbb{L}(0) = (L_\infty)^n_+$ (property $\mathbb{L}.1$, Section (2.2.1)), implying $\mathbb{K}(0,p) = 0$ and $\underline{\bar{\Psi}}^*(u,x) = +\infty$, which in turn implies $\mathbb{L}^*(u) = (L_\infty)^n_+$. If $u \geq 0$ and $\mathbb{L}(u)$ is empty, $\mathbb{K}(u,p) = +\infty$ for $p \in (L_1)^m_+$, implying $\underline{\bar{\Psi}}^*(u,x) = 0$ for $x \in (L_\infty)^n_+$, and thereby $\mathbb{L}^*(u)$ is empty. Thus we need only consider further that $u \geq 0$ and $\mathbb{L}(u)$ is not empty. Let $x^o \in \mathbb{L}(u)$, $u \geq 0$.

From the definition of $\mathbb{K}(u,p)$ it follows that

$$\langle p,x^o \rangle \geqq \mathbb{K}(u,p) \quad \text{for all} \quad p \in (L_1)^n_+ ,$$

and, if $p \in \left\{p \in (L_1)^n_+ : \mathbb{K}(u,p) \geqq 1\right\}$,

$$\langle p,x^o \rangle \geqq \mathbb{K}(u,p) \geqq 1 .$$

Accordingly, from the definition of $\underline{\bar{\Psi}}^*(u,x^o)$ as the infimum of $\langle p,x^o \rangle$ for $\mathbb{K}(u,p) \geqq 1$, it follows that there exists for any positive scalar $\epsilon > 0$ a vector of price functions $p_\epsilon \in \left\{p \in (L_1)^m_+ : \mathbb{K}(u,p) \geqq 1\right\}$ such that

$$\underline{\bar{\Psi}}^*(u,x^o) + \epsilon > \langle p_\epsilon,x^o \rangle \geqq 1 ,$$

whence $\underline{\bar{\Psi}}^*(u,x^o) \geqq 1$ and $x^o \in \mathbb{L}^*(u)$. Thus $\mathbb{L}(u) \subset \mathbb{L}^*(u)$.

The following proposition states the Weak Global Duality between minimal cost functional and distance functional for the dynamic input structure.

Proposition (7.1-2): (Shephard, 1970:a)

If the dynamic input correspondence $u \to \mathbb{L}(u)$ satisfies the weak properties $\mathbb{L}.1, \ldots, \mathbb{L}.6$, in weak form,

$$\underline{\bar{\Psi}}(u,x) \leqq \operatorname*{Inf}_{p} \left\{\langle p,x\rangle : \mathbb{K}(u,p) \geqq 1 , p \in (L_1)^n_+\right\}$$

$$\mathbb{K}(u,p) = \operatorname*{Min}_{x} \left\{\langle p,x\rangle : \underline{\bar{\Psi}}(u,x) \geqq 1 , x \in (L_\infty)^n_+\right\}$$

for $u \in (L_\infty)^m_+$, $x \in (L_\infty)^n_+$, $p \in (L_1)^n_+$.

The second part of the statement of Proposition (7.1-2) follows from the definition (6.1-1) and Proposition (4.2-1). Concerning the first part, there are two cases to eliminate in comparing $\underline{\bar{\Psi}}(u,x)$ and $\underline{\bar{\Psi}}^*(u,x)$.

(a) $\underline{\bar{\Psi}}(u,x) > \underline{\bar{\Psi}}^*(u,x)$, $\underline{\bar{\Psi}}^*(u,x) = 0$

(b) $\underline{\bar{\Psi}}(u,x) > \underline{\bar{\Psi}}^*(u,x)$, $\underline{\bar{\Psi}}^*(u,x) > 0$.

For case (a), let $x^o \in (L_\infty)^n_+$ be a vector of input histories such that $\underline{\bar{\Psi}}^*(u,x^o) = 0$. Then $\underline{\bar{\Psi}}(u,x^o) > 0$. Let $\lambda = [\underline{\bar{\Psi}}(u,x^o)]^{-1}$. Then $\underline{\bar{\Psi}}(u,\lambda x^o) = 1$, implying $(\lambda x^o) \in \mathbb{L}(u) \subset \mathbb{L}^*(u)$, by Proposition (7.1-1). Hence, by the definition (7.1-2)

$$\underline{\bar{\Psi}}^*(u,\lambda x^o) = \lambda\, \underline{\bar{\Psi}}^*(u,x^o) \geqq 1 ,$$

and $\underline{\bar{\Psi}}^*(u,x^o) > 0$, a contradiction. Hence case (a) cannot hold.

For case (b), choose x^o so that $\underline{\bar{\Psi}}^*(u,x^o) > 0$. Then from

$$\lambda \bar{\underline{\Psi}}^{*}(u,x^{o}) \geqq 1$$

and

$$\lambda = [\bar{\underline{\Psi}}(u,x^{o})]^{-1} ,$$

it follows that $\bar{\underline{\Psi}}^{*}(u,x^{o}) \geqq \bar{\underline{\Psi}}(u,x^{o})$, a contradiction. Thus case (b) cannot hold.

For a globally strong duality between minimal cost functional and distance functional for input correspondence, the inequality for the first statement of Proposition (7.1-2) is strengthened to an equality when the map sets $\mathbb{L}(u)$ of vectors of input histories to attain $u \in (L_\infty)_+^m$ are convex, i.e., if

$$x \in \mathbb{L}(u) \text{ , } y \in \mathbb{L}(u) \text{ , } ((1 - \lambda)x + \lambda y) \in \mathbb{L}(u) \quad \text{for} \quad \lambda \in [0,1] ,$$

and axiom $\mathbb{P}.3SS \Longleftrightarrow \mathbb{L}.3SS$ holds, i.e., input rates are freely disposable for the input histories $x \in (L_\infty)_+^n$. Since prices are chosen in $(L_1)_+^n$, a weak* topology is used for $x \in \mathbb{L}(u)$. Then the following proposition may be stated.

Proposition (7.1-3): (Shephard, 1953, 1970:a)

If the dynamic input correspondence $u \to \mathbb{L}(u)$ satisfies $\mathbb{L}$.3SS and $\mathbb{L}(u)$ is convex for $u \in (L_\infty)^m_+$ under a weak* topology for $u \to \mathbb{L}(u)$

$$\underline{\bar{\Psi}}(u,x) = \operatorname*{Inf}_{p} \left\{ \langle p,x \rangle : \mathbb{K}(u,p) \geqq 1 , \ p \in (L_1)^n_+ \right\}$$

$$\mathbb{K}(u,p) = \operatorname*{Min}_{x} \left\{ \langle p,x \rangle : \underline{\bar{\Psi}}(u,x) \geqq 1 , \ x \in (L_\infty)^n_+ \right\},$$

for $u \in (L_\infty)^m_+$, $x \in (L_\infty)^n_+$, $p \in (L_1)^n_+$.

For proof one need only show in addition to that for Proposition (7.1-2) that $\underline{\bar{\Psi}}(u,x) \geqq \underline{\bar{\Psi}}^*(u,x)$, and to this end Proposition (7.1-1) is strengthened to:

Proposition (7.1-4): (Shephard, 1970:a)

$\mathbb{L}(u) = \mathbb{L}^*(u)$ if $u \to \mathbb{L}(u)$ satisfies $\mathbb{L}$.3SS and $\mathbb{L}(u)$ is convex for $u \in (L_\infty)^m_+$ under a weak* topology for $u \to \mathbb{L}(u)$.

Here we need only show beyond Proposition (7.1-1) that $\mathbb{L}^*(u) \subset \mathbb{L}(u)$. If $u = 0$, $\underline{\bar{\Psi}}^*(0,x) = +\infty$ for $x \in (L_\infty)^n_+$ and $\mathbb{L}^*(0) = (L_\infty)^n_+ = \mathbb{L}(0)$. If $u \geq 0$ and $\mathbb{L}(u)$ is empty, $\mathbb{K}(u,p) = +\infty$ for $p \in (L_1)^n_+$, implying $\underline{\bar{\Psi}}^*(u,x) = 0$ and $\mathbb{L}^*(u)$ is likewise empty by the definition (7.1-2). It remains to consider $u \geq 0$, with $\mathbb{L}(u)$ not empty. Now $\mathbb{L}(u)$ not empty, $u \geq 0$, implies $\mathbb{L}^*(u) \supset \mathbb{L}(u)$ also nonempty.

Now suppose $\mathbb{L}^*(u) \not\subset \mathbb{L}(u)$. Then there exists $x^o \in (L_\infty)^n_+$, $x^o \geq 0$ such that $x^o \in \mathbb{L}^*(u)$, but $x^o \notin \mathbb{L}(u)$. Then, using the weak* topology for $(L_\infty)^n_+$, with $\mathbb{L}(u)$ convex and the correspondence $u \to \mathbb{L}(u)$ being taken closed (property $\mathbb{L}$.5) in this topology, and input histories

disposable by $\mathbb{L}$.3SS, by the strict separation theorem (see Berge, 1963), there exists a vector $p \in (L_1)^n_+$, $p \geq 0$, of price functions such that $\langle p,x^o \rangle < \mathbb{K}(u,p)$ and $\langle \hat{p},x^o \rangle < 1$ where

$$\hat{p} := \frac{p}{\mathbb{K}(u,p)} \in \left\{ p \in (L_1)^n_+ : \mathbb{K}(u,p) \geqq 1 \right\} .$$

Since $x^o \in \mathbb{L}^*(u)$, $\bar{\underline{\Psi}}^*(u,x^o) \leqq \langle p,x^o \rangle < 1$, a contradiction. Hence $\mathbb{L}^*(u) \subset \mathbb{L}(u)$ and $\mathbb{L}^*(u) = \mathbb{L}(u)$.

Now, continuing with Proposition (7.1.3), suppose $\bar{\underline{\Psi}}(u,x^o) < \bar{\underline{\Psi}}^*(u,x^o)$ for some $x^o \in (L_\infty)^n_+$. As before we have two cases

(a) $\bar{\underline{\Psi}}^*(u,x^o) > \bar{\underline{\Psi}}(u,x^o) = 0$

(b) $\bar{\underline{\Psi}}^*(u,x^o) > \bar{\underline{\Psi}}(u,x^o) > 0$.

By similar argument to that given for $\bar{\underline{\Psi}}(u,x) > \bar{\underline{\Psi}}^*(u,x^o)$ using the fact that $\mathbb{L}^*(u) = \mathbb{L}(u)$, the two cases are shown to be not possible. Thus $\bar{\underline{\Psi}}^*(u,x) = \bar{\underline{\Psi}}(u,x)$ for $u \in (L_\infty)^m_+$, $x \in (L_\infty)^n_+$.

If one seeks to construct the dynamic input correspondence $u \to \mathbb{L}(u)$ completely from the minimal cost functional, convexity and free disposability of input histories for the map sets $\mathbb{L}(u)$ are required. However for some economic analyses one may be interested only in a given vector $p \geq 0$ of price histories for inputs, so that only input vectors $\tilde{x}(p)$ such that $\langle p , \tilde{x}(p) \rangle = \mathbb{K}(u,p)$, and scalar extensions of the same, are of interest. For this situation where $p \geq 0$, $p \in (L_1)^n_+$, define

(7.1-3) $\mathbb{W}_p(u) := \{\lambda x : x \in \mathbb{L}(u) , \langle p,x \rangle = \mathbb{K}(u,p) , \lambda \in [1,+\infty)\}$.

Then the following proposition holds without recourse to the weak* topology for $u \to \mathbb{L}(u)$.

Proposition (7.1-4): (Färe, 1978:a)

For $p \geq 0$, $p \in (L_1)^n_+$, $u \in (L_\infty)^m_+$ and $\mathbb{L}(u) \neq \emptyset$, and only the weak axioms $\mathbb{L}.1, \ldots, \mathbb{L}.6$ holding,

$$(7.1\text{-}4) \qquad \underline{\bar{\Psi}}(u,x) = \operatorname*{Inf}_{\lambda} \{ \langle \lambda p, x \rangle : \lambda \in [0,+\infty) , \mathbb{K}(u,\lambda p) \geqq 1\}$$

if and only if $x \in \mathbb{W}_p(u)$.

For $\mathbb{W}_p(u) \subset \mathbb{L}(u)$ a distance functional is defined by (see Section 4.2)

$$(7.1\text{-}5) \qquad \underline{\bar{\Psi}}_p(u,x) : = [\text{Inf } \{\lambda : (\lambda x) \in \mathbb{W}_p(u) , \lambda \in [0,+\infty)\}]^{-1} ,$$

and, like Propositions (4.2-1) and (4.2-2),

$$(7.1\text{-}6) \qquad \mathbb{W}_p(u) = \left\{ x \in (L_\infty)^n_+ : \underline{\bar{\Psi}}_p(u,x) \geqq 1 \right\} .$$

$$(7.1\text{-}7) \qquad \text{ISOQ } \mathbb{W}_p(u) = \left\{ x \in (L_\infty)^n_+ : \underline{\bar{\Psi}}_p(u,x) = 1 \right\} .$$

Assume first that $x \in \mathbb{W}_p(u)$. Then $x = \lambda y$, where $y \in \text{ISOQ } \mathbb{W}_p(u)$ implying $y \in \text{ISOQ } \mathbb{L}(u)$, and $\underline{\bar{\Psi}}(u,x) = \underline{\bar{\Psi}}(u,\lambda y) = \lambda \underline{\bar{\Psi}}(u,y) = \lambda$. Also, $\underline{\bar{\Psi}}_p(u,x) = \underline{\bar{\Psi}}_p(u,\lambda y) = \lambda \underline{\bar{\Psi}}_p(u,y) = \lambda$. Now, using (7.1-3),

$$\underline{\bar{\Psi}}_p(u,x) = [\text{Inf } \{\lambda : \langle p, \lambda x \rangle \geqq \mathbb{K}(u,p)\}]^{-1}$$
$$= \frac{\langle p,x \rangle}{\mathbb{K}(u,p)} ,$$

and

$$\underline{\bar{\Psi}}(u,x) = \frac{\langle p,x\rangle}{\mathbb{K}(u,p)} = \operatorname*{Inf}_{\lambda}\{\langle \lambda p,x\rangle : \mathbb{K}(u,\lambda p) \geqq 1 , \lambda \in [0,+\infty)\} .$$

Conversely, let $\underline{\bar{\Psi}}_p^*(u,x)$ denote the right hand side of (7.1-4), and suppose $\underline{\bar{\Psi}}(u,x) = \underline{\bar{\Psi}}_p^*(u,x)$. Define also

$$\mathbb{W}_p^*(u) = \left\{x \in (L_\infty)_+^n : x \in \mathbb{L}(u) , \underline{\bar{\Psi}}_p^*(u,x) \geqq 1\right\} .$$

Then, by an argument analogous to that given in Proposition (7.1-1), it may be shown that $\mathbb{W}_p(u) \subset \mathbb{W}_p^*(u)$. Next let $(\lambda x^o) \in \mathbb{W}_p^*(u)$, and assume $(\lambda x^o) \notin \mathbb{W}_p(u)$. The first implies

$$\underline{\bar{\Psi}}_p^*(u,\lambda x^o) = \frac{\lambda\langle p,x^o\rangle}{\mathbb{K}(u,p)} \geqq 1 ,$$

while the second implies

$$\underline{\bar{\Psi}}_p(u,\lambda x^o) = \frac{\lambda\langle p,x^o\rangle}{\mathbb{K}(u,p)} < 1 ,$$

a contradiction. Thus $\mathbb{W}_p^*(u) \subset \mathbb{W}_p(u)$ and $\mathbb{W}_p^*(u) = \mathbb{W}_p(u)$. By an argument similar to that of Proposition (7.1-3) it may be shown that

$$\underline{\bar{\Psi}}_p^*(u,x) = \underline{\bar{\Psi}}_p(u,x) , x \in (L_\infty)_+^n .$$

Note that when $(\lambda x^o) \notin \mathbb{W}_p(u)$ for $\lambda \in [0,+\infty)$, $\underline{\bar{\Psi}}_p(u,x^o) = \underline{\bar{\Psi}}(u,x^o) = 0$, implying $\mathbb{K}(u,p) = +\infty$ and $\underline{\bar{\Psi}}_p^*(u,x^o) = 0$, and also $(\lambda x^o) \notin \mathbb{W}_p^*(u)$ for $\lambda \in [0,+\infty)$. Finally if $x \in \mathbb{L}(u)$ and $\underline{\bar{\Psi}}_p^*(u,x) = \underline{\bar{\Psi}}(u,x)$, then

$$\underline{\bar{\Psi}}_p(u,x) = \underline{\bar{\Psi}}_p^*(u,x) = \underline{\bar{\Psi}}(u,x) \geqq 1 ,$$

and $x \in \mathbb{W}_p(u)$.

The significance of Proposition (7.1.4) is that when the vector x of input histories belongs to the aureoled subset of cost minimizers of $\mathbb{L}(u) \neq \emptyset$ for a given vector $p \geq 0$ of price histories for the factors, the value of the distance functional of $u \to \mathbb{L}(u)$ for x may be expressed as a partial minimal problem of structure similar to that for the global duality. This partial strong form of the duality holds under the weak axioms for the correspondence.

7.2 Duality Between Maximal Revenue Functional and Distance Functional for Output Correspondence

The analysis for duality between maximal revenue functional $\mathbb{R}(x,r)$ and the distance functional $\Omega(x,u)$ so closely follows that given for the minimal cost functional $\mathbb{K}(u,p)$ and the distance functional $\bar{\underline{\Psi}}(u,x)$, that details need not be repeated here.

A global weak duality between $\mathbb{R}(x,r)$ and $\Omega(x,u)$ is expressed by:

Proposition (7.2-1): (Shephard, 1970:a)

If the dynamic output correspondence $x \to \mathbb{P}(x)$ satisfies $\mathbb{P}.1, \ldots, \mathbb{P}.6$ in weak form:

$$\Omega(x,u) \leqq \operatorname*{Sup}_{r} \left\{ \langle r,u \rangle : \mathbb{R}(x,r) \leqq 1 , r \in (L_1)^m \right\}$$

$$\mathbb{R}(x,r) = \operatorname*{Max}_{u} \left\{ \langle r,u \rangle : \Omega(x,r) \leqq 1 , u \in (L_\infty)^m_+ \right\}$$

for $x \in (L_\infty)^n_+$, $u \in (L_\infty)^m_+$, $r \in (L_1)^m$.

A globally strong form of the duality between $\mathbb{R}(x,r)$ and $\Omega(x,u)$ is stated by

Proposition (7.2-2): (Shephard, 1970:a)

If the dynamic output correspondence satisfies $\mathbb{P}.6SS$ and $\mathbb{P}(x)$ is convex for $x \varepsilon (L_\infty)_+^n$,

$$\Omega(x,u) = \operatorname*{Sup}_r \left\{ \langle r,u \rangle : \mathbb{R}(x,r) \leqq 1 \text{ , } r \varepsilon (L_1)^m \right\}$$

$$\mathbb{R}(x,r) = \operatorname*{Max}_u \left\{ \langle r,u \rangle : \Omega(x,u) \leqq 1 \text{ , } u \varepsilon (L_\infty)_+^m \right\}$$

for $u \varepsilon (L_\infty)_+^m$, $x \varepsilon (L_\infty)_+^n$, $r \varepsilon (L_1)^m$, under the weak* topology for $x \to \mathbb{P}(x)$.

Again, one cannot construct the dynamic output correspondence $x \to \mathbb{P}(x)$ completely from the maximal revenue functional unless the sets $\mathbb{P}(x)$ of vectors of output histories have freely disposable histories and $\mathbb{P}(x)$ is a convex subset of $(L_\infty)_+^m$. However a partial duality in the strong form holds under weak axioms for the correspondence.

Define, for given $r \varepsilon (L_1)^m$, $x \varepsilon (L_\infty)_+^n$, $\mathbb{R}(x,r) > 0$,

(7.2-1) $\mathbb{W}_r(x) := \{\theta u : u \varepsilon \mathbb{P}(x) \text{ , } \langle r,u \rangle = \mathbb{R}(x,r) \text{ , } \theta \varepsilon [0,1]\}$.

Then the following partial duality in the strong form holds:

Proposition (7.2-3): (Färe, 1978:a)

For $r \varepsilon (L_1)^m$, $x \varepsilon (L_\infty)_+^n$ with $\mathbb{R}(x,r) > 0$,

(7.2-2) $\Omega(x,u) = \operatorname*{Sup}_\theta \{ \langle \theta r,x \rangle : \theta \varepsilon [0,+\infty) \text{ , } \mathbb{R}(x,\theta r) \leqq 1\}$

if and only if $u \varepsilon \mathbb{W}_r(x)$.

7.3 Shadow Price Functions for Input and Output Histories

Consider first the case where a vector $x^o \in (L_\infty)^n_+$ of input histories is given and a vector $r^o \in (L_1)^m$ of price histories for outputs is likewise given. With x^o a set $\mathbb{P}(x^o)$ of vectors of output histories, and for any $u \in \mathbb{P}(x^o)$, $\langle r^o,u \rangle$ is the value of that result from the use of x . The maximal value of $\langle r^o,u \rangle$ for $u \in \mathbb{P}(x^o)$ is given by $\mathbb{R}(x^o,r^o)$ for some $u^* \in \mathbb{P}(x^o)$, depending upon x^o and r^o . The value $\mathbb{R}(x^o,r^o)$ represents the best result possible from the use of x^o . One seeks a vector $p^s \in (L_1)^n_+$ of shadow price histories for the input histories of x^o which reflects their contribution (technical importance) in attaining $\mathbb{R}(x^o,r^o)$. It is clear that the shadow price history vector p^s may depend upon x^o and r^o .

For the construction of the vector p^s of shadow price histories, suppose $\mathbb{R}(x^o,r^o) > 0$, otherwise an imputation of price histories p^s is meaningless. In some way the valuations of the components of x^o sought are the minimal price histories p^s_i possible subject to the constraint that the so imputed value $\langle p^s,x^o \rangle$ is at least as large as the maximal value $\mathbb{R}(x^o,r^o)$ attainable with x^o .

Consider any feasible vector pair, $u \in \mathbb{P}(x^o)$. The distance functional for $u \to \mathbb{L}(u)$ satisfies (by Proposition (7.1-2))

$$(7.3\text{-}1) \qquad \underline{\bar{\Psi}}(u,x^o) \leqq \operatorname*{Inf}_p \left\{ \langle p,x^o \rangle : \mathbb{K}(u,p) \geqq 1 , p \in (L_1)^n_+ \right\}$$

with $u^*(x^o,r^o)$ yielding

$$(7.3\text{-}2) \qquad \mathbb{R}(x^o,r^o) = \operatorname*{Max}_u \left\{ \langle r^o,u \rangle : u \in \mathbb{P}(x^o) \right\} := \langle r^o,u^* \rangle .$$

The inequality (7.3-1) may be rewritten

$$(7.3\text{-}3)\quad \bar{\underline{\Psi}}(u,x^o) \leqq \frac{1}{\mathbb{R}(x^o,r^o)} \operatorname*{Inf}_{p} \left\{ \langle p,x^o \rangle : \mathbb{K}(u,p) \geqq \mathbb{R}(x^o,r^o) \ , \ p \ \varepsilon \ (L_1)^m_+ \right\} .$$

Then since $u^* \ \varepsilon \ \mathbb{P}(x^o)$, implying $x^o \ \varepsilon \ \mathbb{L}(u^*)$, $\bar{\underline{\Psi}}(u,x^o) \geqq 1$, and

$$\operatorname*{Inf}_{p} \left\{ \langle p,x^o \rangle : \mathbb{K}(u^*,p) \geqq \mathbb{R}(x^o,r^o) \ , \ p \ \varepsilon \ (L_1)^m_+ \right\} \geqq \langle r^o,u^* \rangle .$$

Accordingly, the following proposition holds:

<u>Proposition (7.3-1)</u>: (Shephard, 1970:a)

For $x^o \ \varepsilon \ (L_\infty)^n_+$, $r^o \ \varepsilon \ (L_1)^m$ given with $\mathbb{R}(x^o,r^o) > 0$, a vector p^s solving the problem

$$(7.3\text{-}4)\quad \operatorname*{Inf}_{p} \left\{ \langle p \cdot x^o \rangle : \mathbb{K}(u^*,p) \geqq \mathbb{R}(x^o,r^o) \ , \ p \ \varepsilon \ (L_1)^m_+ \right\}$$

where $\langle r^o,u^* \rangle = \mathbb{R}(x^o,r^o)$, is a minimal vector of shadow price histories *for* x^o *satisfying*

$$(7.3\text{-}5)\qquad \langle p^s,x^o \rangle \geqq \langle r^o,u^* \rangle = \mathbb{R}(x^o,r^o) .$$

In case $x^o \ \varepsilon$ ISOQ $\mathbb{L}(u^*)$ and the globally strong dual expression for $\bar{\underline{\Psi}}(u,x)$ holds, the inequality sign in (7.3-5) may be changed to an equal sign.

In a similar way a vector r^s of shadow price histories may be constructed for a given vector $u^o \ \varepsilon \ (L_\infty)^m_+$ when a vector $p^o \ \varepsilon \ (L_1)^n_+$ of price histories is given for inputs, where $\mathbb{L}(u^o) \neq \emptyset$, $u^o \geq 0$ and $\mathbb{K}(u^o,p^o) > 0$.

Let $x^* \in \mathbb{L}(u^o)$ yield $\mathbb{K}(u^o,p^o)$ and

$$1 \geqq \Omega(x^*,u^o) \geqq \frac{1}{\mathbb{K}(u^o,p^o)} \operatorname*{Sup}_r \left\{\langle r,u^o \rangle : \mathbb{R}(x^*,r) \leqq \mathbb{K}(u^o,p^o) \ , \ r \in (L_1)^m \right\} .$$

Then the following proposition is evident.

<u>Proposition (7.3-2)</u>: (Shephard, 1970:a)

For $u^o \in (L_\infty)^m_+$, $p^o \in (L_1)^m_+$, $\mathbb{L}(u^o) \neq \emptyset$ and $\mathbb{K}(u^o,p^o) > 0$, a vector r^s solving the problem

$$\text{(7.3-6)} \qquad \operatorname*{Sup}_r \left\{\langle r,u^o \rangle : \mathbb{R}(x^*,r) \leqq \mathbb{K}(u^o,p^o) \ , \ r \in (L_1)^m \right\} ,$$

where $\langle p^o,x^* \rangle = \mathbb{K}(u^o,p^o)$, is a maximal vector of shadow price histories for u^o satisfying

$$\text{(7.3-7)} \qquad \langle r^s,u^o \rangle \leqq \langle p^o,x^* \rangle = \mathbb{K}(u^o,p^o) .$$

In case the globally strong dual expression for $\Omega(x,u)$ holds and $u^o \in \text{ISOQ } \mathbb{P}(x^*)$, the inequality sign of (7.3-7) may be changed to an equality sign.

Vectors of shadow price histories may be determined for both input and output histories for any technically feasible pair $x^o \in (L_\infty)^n_+$, $u^o \in (L\ )^m_+$, i.e., $u^o \in \mathbb{P}(x^o) \Longleftrightarrow x^o \in \mathbb{L}(u^o)$. By the globally weak dual expressions for $\bar{\underline{\Psi}}(u^o,x^o)$ and $\Omega(x^o,u^o)$, and Propositions (4.1-1) and (4.2-1),

$$\begin{aligned}(7.3\text{-}8)\qquad &\operatorname*{Inf}_{p}\left\{\langle p,x^{o}\rangle : \mathbb{K}(u^{o},p) \geqq 1 \ , \ p \ \varepsilon \ (L_1)^{n}_{+}\right\} \\ &\leqq \operatorname*{Sup}_{r}\left\{\langle r,u^{o}\rangle : \mathbb{R}(x^{o},r) \leqq 1 \ , \ r \ \varepsilon \ (L_1)^{m}\right\} .\end{aligned}$$

Accordingly the following proposition holds:

<u>Proposition (7.3-3)</u>: (Shephard, 1970:a)

For $u^{o} \ \varepsilon \ (L_{\infty})^{m}_{+}$, $x^{o} \ \varepsilon \ (L_{\infty})^{n}_{+}$, $u^{o} \ \varepsilon \ \mathbb{P}(x^{o})$ vectors p^{s} , r^{s} respectively solving the problems

$$(7.3\text{-}9)\qquad \operatorname*{Inf}_{p}\left\{\langle p,x^{o}\rangle : \mathbb{K}(u^{o},p) \geqq 1 \ , \ p \ \varepsilon \ (L_1)^{m}_{+}\right\}$$

$$(7.3\text{-}10)\qquad \operatorname*{Sup}_{r}\left\{\langle r,u^{o}\rangle : \mathbb{R}(x^{o},r) \leqq 1 \ , \ r \ \varepsilon \ (L_1)^{m}\right\}$$

are minimal and maximal vectors of shadow price histories for x^{o} and u^{o} respectively, satisfying

$$(7.3\text{-}11)\qquad \langle r^{s},u^{o}\rangle \leqq \langle p^{s},x^{o}\rangle \ .$$

The inequality (7.3-11) becomes an equality if $x^{o} \ \varepsilon$ ISOQ $\mathbb{L}(u^{o})$, $u^{o} \ \varepsilon$ ISOQ $\mathbb{P}(x^{o})$, and the globally strong dual expressions hold for $\Omega(x,u)$ and $\bar{\underline{\Psi}}(u,x)$.

When the dynamic input or output structure has ray homothetic form, the shadow price history mixes are not altered by scaling of inputs and outputs. If the dynamic output correspondence is ray homothetic,

$$\mathbb{R}(\lambda x^{o},r) = \frac{F(\mathbb{H}(\lambda x^{o}))}{F\left(\mathbb{H}\left(\frac{x^{o}}{||x^{o}||}\right)\right)} \cdot \mathbb{R}\left(\frac{x^{o}}{||x^{o}||} \ , \ r\right)$$

for $\lambda \varepsilon (0,+\infty)$. Then the minimal problem defining p^s for given $\lambda x^o \varepsilon (L_\infty)^n_+$, $r^o \varepsilon (L_1)^m$, with $\mathbb{R}\left(\frac{x^o}{||x^o||}, r\right) > 0$, and $\lambda \varepsilon (0,+\infty)$, becomes

$$\operatorname*{Inf}_{p}\left\{\langle p,\lambda x^o\rangle : \mathbb{K}(u^*,p) \geqq \frac{F(\mathbb{H}(\lambda x^o))}{F\left(\mathbb{H}\left(\frac{x^o}{||x^o||}\right)\right)} \cdot \mathbb{R}\left(\frac{x^o}{||x^o||}, r^o\right), p \varepsilon (L_1)^n_+\right\}$$

$$= \lambda \cdot \frac{F(\mathbb{H}(\lambda x^o))}{F\left(\mathbb{H}\left(\frac{x^o}{||x^o||}\right)\right)} \operatorname*{Inf}_{p}\left\{\langle p,x^o\rangle : \mathbb{K}(u^*,p) \geqq \mathbb{R}\left(\frac{x^o}{||x^o||}, r^o\right), p \varepsilon (L_1)^n_+\right\} .$$

In the case where the dynamic input structure is ray homothetic, $u^o \varepsilon (L_\infty)^m_+$, $p^o \varepsilon (L_1)^n_+$, $\mathbb{L}(u^o) \neq \emptyset$ and $\mathbb{K}(u^o,p^o) > 0$, the maximal problem defining r^s for (θu^o) , $\theta \varepsilon (0,+\infty)$, becomes

$$\operatorname*{Sup}_{r}\{\langle r,\theta u^o\rangle : \mathbb{R}(x^*,r) \leqq \mathbb{K}(u^o,p^o)\}$$

$$= \theta \frac{G(\mathbb{J}(\theta u^o))}{G\left(\mathbb{J}\left(\frac{u^o}{||u^o||}\right)\right)} \cdot \operatorname*{Sup}_{r}\left\{\langle r,u^o\rangle : \mathbb{R}(x^*,r) \leqq \mathbb{K}\left(\frac{u^o}{||u^o||}, p^o\right), r \varepsilon (L_1)^m\right\} .$$

When both p^s and r^s are simultaneously determined for a feasible pair, and either x^o or u^o is scaled so that $(\lambda x^o) \varepsilon \mathbb{L}(\theta u^o)$, the shadow price mixes are likewise independent of the scaling factors λ , θ .

Thus the following proposition holds:

<u>Proposition (7.3-4)</u>:

If the dynamic output structure (input structure) is ray homothetic, the relative price histories of the shadow price vector p^s (r^s) and

independent of scaling of x^o (u^o) , when a price history vector r^o (p^o) is given. Feasible scaled vectors of input and output histories have shadow price history mixes independent of the scaling factors.

CHAPTER 8

INDEX FUNCTIONS FOR PRODUCTION THEORY

In the static theory of the price index, prices at two points of time for a given bundle of goods and services are compared by the ratio of the implied cost of the bundle at these two points of time. In case the bundle represents a collection of consumption goods and a representative utility function is used to optimize a vector of input rates of these goods, a "cost of living index" is formulated by comparing the minimal cost of achieving a given level of utility (satisfaction) at the two points of time, which in general is not independent of the level of satisfaction selected.

For the dynamic theory of production it is useful to formulate index functions which represent at a time $T \varepsilon (0,+\infty)$, a comparison of two aggregate levels, cumulatively, of prices and quantities for both inputs and outputs, in such a way that the index functions so developed serve a dynamic macroeconomic expression of the theory. Since the use of value ratios to define index numbers is of long tradition in economics, this approach will be used here.

8.1 Price and Quantity Indices for Inputs[†]

Let $p^1 \varepsilon (L_1)^n_+$, $p^0 \varepsilon (L_1)^n_+$, $p^0 \geq 0$ be two vectors of price histories for vectors $x \varepsilon (L_\infty)^n_+$ of input histories, and let $u \varepsilon (L_\infty)^n_+$, $u \geq 0$, $\mathbb{L}(u) \neq \emptyset$, be a reference vector of output histories, to which input histories are related for definition of price indices.

[†]The contents of this chapter are modifications of (Shephard, 1978), and (Färe, 1978:b).

At a time $T \varepsilon (0,+\infty)$ for definition of the index of price level comparison for the two price history vectors p^1 and p^0, let

$$u \cdot [0,T] := (v : v(t) = u(t) \ , \ t \varepsilon [0,T] \ , \ v(t) = 0 \ , \ t > T) \ .$$

Then an index of p^1 relative to p^0 over $[0,T]$ is defined for $\mathbb{K}_T(u,p^0) > 0$, by

$$\text{(8.1-1)} \qquad \pi_T(p^1,p^0 \mid u) := \frac{\mathbb{K}_T(u,p^1)}{\mathbb{K}_T(u,p^0)} \ , \ T \varepsilon (0,+\infty) \ ,$$

where

$$\text{(8.1-2)} \qquad \mathbb{K}_T(u,p) := \operatorname*{Min}_x \{\langle p,x \rangle : x \varepsilon \mathbb{L}(u \cdot [0,T])\} \ .$$

An index function for comparing p^1 and p^0 cumulatively for T ranging over $(0,+\infty)$ with $p^0 \varepsilon \left\{p \varepsilon (L_1)^m_+ : \mathbb{K}_T(u,p^0) > 0 \ , \ T \varepsilon (0,+\infty)\right\}$, is defined by

$$\text{(8.1-3)} \qquad \begin{aligned} &\pi : (p^1,p^0) \varepsilon (L_1)^n_+ \times (L_1)^n_+ \to \pi(p^1,p^0 \mid u) \varepsilon (L_\infty) \ , \\ &\pi(p^1,p^0 \mid u,T) := \pi_T(p^1,p^0 \mid u) \ , \ T \varepsilon (0,+\infty) \ . \end{aligned}$$

We may assign

$$\text{(8.1-4)} \qquad \lim_{T\to\infty} \pi_T(p^1,p^0 \mid u) = \frac{\mathbb{K}(u,p^1)}{\mathbb{K}(u,p^0)} \ ,$$

since $\mathbb{L}(u \cdot [0,T]) \subset \mathbb{L}(u \cdot [0,T']) \subset \mathbb{L}(u)$ for all $0 < T \leqq T' < +\infty$ and $\mathbb{K}_T(u,p)$ is monotone nondecreasing in T. Thus, interpret (8.1-4) as an infinite horizon comparison of p^1 and p^0.

Turning now to the formulation of an index function for comparing aggregate levels of two vectors $x^1 \in (L_\infty)^n_+$, $x^0 \in (L_\infty)^n_+$ of input histories, cumulatively, at time $T \in (0,+\infty)$, let $r \in (L_1)^m_+$ denote a reference price vector to value outputs derivable from x^1 and x^0 . For comparison of x^1 and x^0 , let r and the base vector x^0 be chosen so that $\mathbb{R}_T(x^0,r) > 0$ for $T \in (0,+\infty)$. Then an index of x^1 relative to x^0 over $[0,T]$ is defined by

$$(8.1\text{-}5) \qquad I_T(x^1,x^0 \mid r) := \frac{G(\hat{\mathbb{R}}_T(x^1,r))}{G(\hat{\mathbb{R}}_T(x^0,r))}$$

where $\hat{\mathbb{R}}_T(x,r)$ is a "standardized" value of

$$(8.1\text{-}6) \qquad \mathbb{R}_T(x,r) := \underset{u}{\text{Max}}\{\langle r,u\rangle : u \in \mathbb{P}(x \cdot [0,T]\} .$$

The standardized value $\hat{\mathbb{R}}_T(x,r)$ of the maximal revenue functional $\mathbb{R}_T(x,r)$ is defined by

$$(8.1\text{-}7) \qquad \hat{\mathbb{R}}_T(x,r) := \frac{\mathbb{R}_T(x,r)}{\underset{u}{\text{Max}}\{\langle r,u\rangle : u \in A \cdot [0,T]\}} ,$$

where A is some appropriately chosen representative subset of output vectors, $A \subset (L_\infty)^m_+$, the maximal revenue from which under the price history vector r serves as a "price level deflator." The function $G(\cdot)$ is any nonnegative, nondecreasing, transformation of "standardized" maximal revenue over $[0,T]$, with $G(0) = 0$ and $G(v) \to +\infty$ for $v \to +\infty$.

The "standardized" values of maximal revenue $\hat{\mathbb{R}}_T(x^1,r)$, $\hat{\mathbb{R}}_T(x^0,r)$ are "real" measures of x^1 and x^2 over $[0,T]$, but there is no absolute unit for their values. Hence the monotone transformation $\hat{\mathbb{R}}_T \in R_+ \to G(\hat{\mathbb{R}}_T) \in R_+$ is used to permit adjustment of the value ratio so that the product of the index of price and quantity over $[0,T]$ may equal the ratio of the minimal cost of obtaining two vectors u^1 and u^0 over $[0,T]$, when the input histories are chosen to minimize cost over $[0,T]$ of obtaining u^1 and u^0 .

The index function for comparing the aggregate levels of x^1 and x^0 cumulatively for T ranging over $(0,+\infty)$, with x^0 and r chosen so that $\mathbb{R}_T(x^0,r) > 0$ for $T \in (0,+\infty)$, is defined by:

$$\begin{aligned} &I : (x^1,x^0) \in (L_\infty)^n_+ \times (L_\infty)^n_+ \to I(x^1,x^0 \mid r) \in (L_\infty)_+ , \\ (8.1\text{-}8)\quad &I(x^1,x^0 \mid r,T) := I_T(x^1,x^0 \mid r) , \ T \in (0,+\infty) . \end{aligned}$$

Similar to (8.1-4) we may assign

$$\lim_{T\to\infty} I_T(x^1,x^0 \mid r) = \frac{G(\hat{\mathbb{R}}(x^1,r))}{G(\hat{\mathbb{R}}(x^0,r))}$$

as an infinite horizon comparison.

8.2 Price and Quantity Indices for Outputs

Similar to the formulations of Section (8.1), let $r^1 \epsilon (L_1)^m$, $r^0 \epsilon (L_1)^m$, $r^0 \nleq 0$, be two vectors of price histories for vectors $u \epsilon (L_\infty)^m_+$ of output histories, and let $x \epsilon (L_\infty)^n_+$ be a reference vector of input histories from which outputs are derived.

For x and r^0 such that $\mathbb{R}_T(x,r^0) > 0$ for $T \epsilon (0,+\infty)$, the index function for cumulatively comparing r^1 and r^0 as T ranges over $(0,+\infty)$ is defined by:

(8.2-1)
$$\rho : (r^1,r^0) \epsilon (L_1)^m_+ \times (L_1)^m_+ \rightarrow \rho(r^1,r^0 \mid x) \epsilon (L_\infty)_+$$
$$\rho(r^1,r^0 \mid x,T) := \rho_T(r^1,r^0 \mid x) , \ T \epsilon (0,+\infty)$$
$$\rho_T(r^1,r^0 \mid x) := \frac{\mathbb{R}_T(x,r^1)}{\mathbb{R}_T(x,r^0)} .$$

In the case of an index function for output histories, let $u^1 \epsilon (L_\infty)^m_+$, $u^0 \epsilon (L_\infty)^m_+$, $u^0 \geq 0$, $\mathbb{L}(u^0) \neq \emptyset$ be two vectors of output histories for comparison over $[0,T]$ as T ranges over $(0,+\infty)$. A reference price vector $p \epsilon (L_1)^m_+$ is chosen so that $\mathbb{K}_T(u^0,p) > 0$ for $T \epsilon (0,+\infty)$. Then the index function comparing u^1 to u^0 is

(8.2-2)
$$O : (u^1,u^0) \epsilon (L_\infty)^m_+ \times (L_\infty)^m_+ \rightarrow O(u^1,u^0 \mid p) \epsilon (L_\infty)_+$$
$$O(u^1,u^0 \mid p,T) := O_T(u^1,u^0 \mid p) , \ T \epsilon (0,+\infty)$$
$$O_T(u^1,u^0 \mid p) := \frac{F(\hat{\mathbb{K}}_T(u^1,p))}{F(\hat{\mathbb{K}}_T(u^0,p))}$$

in which the standardized minimal cost function $\hat{\mathbb{K}}_T(u,p)$ is

$$\hat{\mathbb{K}}_T(u,p) := \frac{\mathbb{K}_T(u,p)}{\underset{x}{\text{Min}}\ \{\langle p,x\rangle : x \in B \cdot [0,T]\}}$$

where B is some appropriately chosen representative subset of input vectors, $B \subset (L_\infty)^n_+$, over which the minimal cost serves as a "price level deflator" for the minimal cost functional. The function $F : \mathbb{K}_T \in R_+ \rightarrow F(\mathbb{K}_T) \in R_+$ is a nonnegative, nondecreasing, transformation of the standardized minimal costs over $[0,T]$, justified as in the case of comparing two input vectors, with $F(0) = 0$ and $F(v) \rightarrow +\infty$ for $v \rightarrow +\infty$.

As before we may assign

$$\lim_{T\to\infty} \rho_T(r^1,r^0 \mid x) = \frac{\mathbb{R}(x^1,r)}{\mathbb{R}(x^0,r)}\ ,\ \lim_{T\to\infty} \bar{O}_T(u^1,u^0 \mid p) = \frac{F(\hat{\mathbb{K}}(u^1,p))}{F(\hat{\mathbb{K}}(u^0,p))}$$

as infinite horizon comparisons.

In the case of a steady state model, when it exists, see Section 2.6, one may treat $u \in R^m_+$, $x \in R^n_+$, $p \in R^n_+$, $r \in R^m$, and replace

$$\mathbb{K}_T(u,p) \text{ by } T \cdot Q(u,p)$$

$$\mathbb{R}_T(x,r) \text{ by } T \cdot R(x,r)$$

$$\hat{\mathbb{K}}_T(u,p) \text{ by } T \cdot \hat{Q}(u,p)$$

$$\hat{\mathbb{R}}_T(u,p) \text{ by } T \cdot \hat{R}(u,p)$$

where

$$Q(u,p) = \underset{x}{\text{Min}} \left\{ p \cdot x \mid x \in R^n_+ ,\ x \in L(u) \right\}$$

$$R(x,r) = \underset{u}{\text{Max}} \left\{ r \cdot u \mid u \in R^m_+ ,\ u \in P(x) \right\}$$

and $\hat{Q}(u,p)$, $\hat{R}(x,r)$ are price deflated values of $Q(u,p)$ and $R(x,r)$. Then the index functions reduce to indices expressed by

$$\pi(p^1,p^0 \mid u) := \frac{Q(u,p^1)}{Q(u,p^0)} , \quad I(x^1,x^0 \mid r) := \frac{G(\hat{R}(x^1,r))}{G(\hat{R}(x^0,r))}$$

$$\rho(r^1,r^0 \mid x) := \frac{R(x,r^1)}{R(x,r^0)} , \quad O(u^1,u^0 \mid p) := \frac{F(\hat{Q}(u^1,p))}{F(\hat{Q}(u^0,p))} .$$

8.3 Index Functions for Homothetic Dynamic Production Structures

Consider first the case of inversely related globally homothetic dynamic structure (see Section 5.1), where

$$\mathbb{P}(x) = F(\Phi(x))\cdot\left\{u \in (L_\infty)^m_+ : \mathbb{F}(u) \leqq 1\right\} , \; x \in (L_\infty)^n_+$$

$$\mathbb{L}(u) = F^{-1}(\mathbb{F}(u)) \cdot \left\{x \in (L_\infty)^n_+ : \Phi(x) \geqq 1\right\} , \; u \in (L_\infty)^m_+ .$$

For a bounded interval $[0,T]$, consider $u \cdot [0,T]$ for $u \in (L_\infty)^m_+$. The minimal cost functional for $u \cdot [0,T]$ becomes,

$$\mathbb{K}_T(u,p) := \underset{x}{\text{Min}} \left\{\langle p,x \rangle : \Phi(x) \geqq F^{-1}(\mathbb{F}(u \cdot [0,T])) , x \in (L_\infty)^n_+ \cdot [0,T]\right\} ,$$

and

$$\text{(8.3-1)} \qquad \mathbb{K}_T(u,p) = F^{-1}(\mathbb{F}(u \cdot [0,T])) \cdot \mathbb{M}_T(p) ,$$

where

$$\text{(8.3-2)} \qquad \mathbb{M}_T(p) := \underset{x}{\text{Min}} \left\{\langle p,x \rangle : \Phi(x) \geqq 1 , x \in (L_\infty)^n_+ \cdot [0,T]\right\}$$

is a minimal cost functional over $[0,T]$ for input vectors constrained to a "standard" set B, defined by

$$B := \left\{x \in (L_\infty)^n_+ : \Phi(x) \geqq 1\right\} . \tag{8.3-3}$$

The maximal revenue function $\mathbb{R}_T(x,r)$ for $x \cdot [0,T]$ becomes

$$\mathbb{R}_T(x,r) = \operatorname*{Max}_u \left\{\langle r,u\rangle : \Psi(u) \leqq F(\Phi(x \cdot [0,T])) , u \in (L_\infty)^m_+ \cdot [0,T]\right\}$$

and

$$\mathbb{R}_T(x,r) = F(\Phi(x \cdot [0,T])) \cdot \mathbb{N}_T(r) \tag{8.3-4}$$

where

$$\mathbb{N}_T(r) := \operatorname*{Max}_u \left\{\langle r,u\rangle : \Psi(u) \leqq 1 , u \in (L_\infty)^m_+ \cdot [0,T]\right\} \tag{8.3-5}$$

is a maximal revenue functional over $[0,T]$ for output vectors constrained to a "standard" set A defined by

$$A := \{u \in (L_\infty) : \Psi(u) \leqq 1\} . \tag{8.3-6}$$

As notation, let $\Psi_T(u)$ denote $\Psi(u \cdot [0,T]))$, and $\Phi_T(x)$ denote $\Phi(x \cdot [0,T])$. Then for a cost minimizing vector $x^* \in (L_\infty)^n_+ \cdot [0,T]$ yielding $\mathbb{K}_T(u,p)$ one has, (see definition preceding (8.3-1)), due to the homogeneity of $\Phi(x)$, that

$$\Phi_T(x^*) = F^{-1}(\Psi_T(u)) . \tag{8.3-7}$$

Similarly, the revenue maximizing vector $u^* \in (L_\infty)^m_+ \cdot [0,T]$ yielding $\mathbb{R}_T(x,r)$ satisfies

(8.3-8) $$\mathbb{F}_T(u^*) = F(\Phi_T(x)) .$$

Accordingly the index functions of Sections (8.1) and (8.2) take simplified forms. First, using (8.3-1),

(8.3-9) $$\pi_T(p^1,p^0 \mid u) = \frac{\mathbb{M}_T(p^1)}{\mathbb{M}_T(p^0)} , \ T \ \varepsilon \ (0,+\infty)$$

independently of the reference vector u . The index comparing x^1 to x^0 over $[0,T]$ becomes

(8.3-10) $$I_T(x^1,x^0 \mid r) = \frac{F^{-1}\left(\dfrac{\mathbb{R}_T(x^1,r)}{\mathbb{N}_T(r)}\right)}{F^{-1}\left(\dfrac{\mathbb{R}_T(x^0,r)}{\mathbb{N}_T(r)}\right)} = \frac{\Phi_T(x^1)}{\Phi_T(x^0)} , \ T \ \varepsilon \ (0,+\infty)$$

using (8.3-4), independent of the reference vector r . From (8.3-4)

(8.3-11) $$\rho_T(r^1,r^0 \mid x) = \frac{\mathbb{N}_T(r^1)}{\mathbb{N}_T(r^0)} , \ T \ \varepsilon \ (0,+\infty)$$

and the index is independent of the reference vector x .

For the last index comparing u^1 and u^0 , use (8.3-1) to find

(8.3-12) $$O_T(u^1,u^0 \mid p) = \frac{F\left(\dfrac{\mathbb{K}_T(u^1,p)}{\mathbb{M}_T(p)}\right)}{F\left(\dfrac{\mathbb{K}_T(u^0,p)}{\mathbb{M}_T(p)}\right)} = \frac{\mathbb{F}_T(u^1)}{\mathbb{F}_T(u^0)} , \ T \ \varepsilon \ (0,+\infty)$$

and again the index is independent of the reference vector p .

In (8.3-9) and (8.3-11) $\mathbb{M}_T(p)$ and $\mathbb{N}_T(r)$, respectively, serve as aggregate measures of price levels for "price deflation" at the time $T \varepsilon (0,+\infty)$.

For a steady state model the four index functions reduce to index numbers

$$\pi(p^1,p^0) = \frac{M(p^1)}{M(p^0)}$$

$$I(x^1,x^0) = \frac{\phi(x^1)}{\phi(x^0)}$$

$$\rho(r^1,r^0) = \frac{N(r^1)}{N(r^0)}$$

$$0(u^1,u^0) = \frac{f(u^1)}{f(u^0)}$$

where x^1 , x^0 , p^1 , $p^0 \varepsilon R^n_+$, u^1 , $u^0 \varepsilon R^m_+$, r^1 , $r^0 \varepsilon R^m$, $\phi(\cdot)$ and $f(\cdot)$ are real valued functions

$$\underline{\bar{\Phi}} : x \varepsilon R^n_+ \rightarrow \underline{\bar{\Phi}}(x) \varepsilon R_+$$

$$f : u \varepsilon R^m_+ \rightarrow f(u) \varepsilon R_+ ,$$

and

$$M(p) = \operatorname*{Min}_x \left\{p \cdot x \mid \phi(x) \geqq 1 , x \varepsilon R^n_+\right\}$$

$$N(r) = \operatorname*{Max}_u \left\{r \cdot u \mid f(u) \leqq 1 , u \varepsilon R^m_+\right\} .$$

It is of some interest whether the index functions satisfy at each time $T \varepsilon (0,+\infty)$ the usual "Tests" or properties required of such indices. The most important of these "Tests" is, for the theory of production, that

$$(8.3\text{-}13) \qquad \frac{\mathbb{K}_T(u^1,p^1)}{\mathbb{K}_T(u^0,p^0)} = \pi_T(p^1,p^0 \mid u) \cdot I((x^1)^*,(x^0)^* \mid r)$$

i.e., the ratio of the minimal cost over $[0,T]$ of obtaining u^1 at price histories p^1 for factors to the minimal cost over $[0,T]$ of obtaining u^0 at price histories p^0 for factors, must equal the product of the price index function value at T for factor prices and the quantity index function value at T for inputs evaluated for cost minimizing input histories $(x^1)^*$, $(x^0)^*$ to obtain u^1 and u^0 respectively. For the inversely related globally homothetic production structures,

$$\frac{\mathbb{K}_T(u^1,p^1)}{\mathbb{K}_T(u^0,p^0)} = \frac{F^{-1}(\mathbb{f}_T(u^1)) \cdot \mathbb{M}_T(p^1)}{F^{-1}(\mathbb{f}_T(u^0)) \cdot \mathbb{M}_T(p^0)} .$$

Using (8.3-7) and (8.3-8)

$$\frac{\mathbb{K}_T(u^1,p^1)}{\mathbb{K}_T(u^0,p^0)} = \frac{\mathbb{M}_T(p^1)}{\mathbb{M}_T(p^0)} \cdot \frac{\Phi_T((x^1)^*)}{\Phi_T((x^0)^*)}$$

$$= \pi_T(p^1,p^0 \mid u) \cdot I_T((x^1)^*,(x^0)^* \mid r)) ,$$

and the "test" is satisfied. Ordinarily, in the static theory of index numbers, Equation (8.3-12) is required for any value ratios, i.e.,

$$\frac{\langle u^1 , p^1 \rangle}{\langle u^0 , p^0 \rangle} = \pi(p^1,p^0) \cdot I(x^1,x^0) ,$$

but our interest here is in cost minimal ratios and for such ratios cost

minimizing values must be used for x^1 and x^0. Another way of putting the matter is that the "test" is required to hold only for comparing minimal costs over intervals $[0,T]$, $T \varepsilon (0,+\infty)$, that is for output history vectors u^1, u^0 and two factor price history vectors p^1, p^0, the test is applied only for "realizable cost minimal factor demand" histories.

Similarly, one may verify that

$$\frac{\mathbb{R}_T(x^1,r^1)}{\mathbb{R}_T(x^0,r^0)} = \rho_T(r^1,r^0 \mid x) \cdot 0_T(u^1,u^0 \mid p)$$

$$= \frac{\mathbb{N}_T(r^1)}{\mathbb{N}_T(r^0)} \cdot \frac{\mathbb{ff}_T((u^1)^*)}{\mathbb{ff}_T((u^0)^*)}$$

where $(u^1)^*$, $(u^0)^*$ are respectively revenue maximizing output histories for $\mathbb{R}_T(x^1,r^1)$ and $\mathbb{R}_T(x^0,r^0)$ respectively.

Another property for the values of the index functions which ordinarily would be required is linear homogeneity, i.e., for example

$$\pi_T(\lambda p^1,p^0 \mid u) = \lambda \pi_T(p^1,p^0 \mid u) \ .$$

The general definitions (8.1-1) and (8.2-1) have this property while (8.1-5) and (8.2-2) do not. However, for inversely related globally homothetic production structures, all four index functions have the linear homogeneity property for each $T \varepsilon (0,+\infty)$.

The time reversal and transitive "tests" are satisfied in the general definition by all four index functions at any time $T \varepsilon (0,+\infty)$, i.e., by (8.1-1), (8.1-5), (8.2-1) and (8.2-2). For example, regarding (8.1-1)

$$\pi_T(p^1,p^0 \mid u) \cdot \pi_T(p^0,p^1 \mid u) = \frac{\mathbb{K}_T(u,p^1)}{\mathbb{K}_T(u,p^0)} \cdot \frac{\mathbb{K}_T(u,p^0)}{\mathbb{K}_T(u,p^1)} = 1 ,$$

$$\pi_T(p^2,p^1 \mid u) \cdot \pi_T(p^1,p^0 \mid u) = \frac{\mathbb{K}_T(u,p^2)}{\mathbb{K}_T(u,p^1)} \cdot \frac{\mathbb{K}_T(u,p^1)}{\mathbb{K}_T(u,p^0)} = \frac{\mathbb{K}_T(u,p^2)}{\mathbb{K}_T(u,p^0)} .$$

A dimensional change in the unit of money or in the unit of any physical quantity does not affect the values of the four indices as defined by (8.3-8), (8.3-9), (8.3-10) and (8.3-11) for inversely related globally homothetic structures.

If the specialization of inversely related globally homothetic structures is relaxed to one of merely both input and output structures being homothetic, i.e.,

$$\mathbb{P}(x) = F(\mathbb{H}(x))\left\{u \in (L_\infty)^m_+ : \mathbb{F}(u) \leqq 1\right\} , \; x \in (L_\infty)^n_+$$

$$\mathbb{L}(u) = G(\mathbb{J}(u))\left\{x \in (L_\infty)^n_+ : \Phi(x) \geqq 1\right\} , \; u \in (L_\infty)^m_+ ,$$

then the minimal cost and maximal revenue functionals become

(8.3-14) $$\mathbb{K}_T(u,p) = G(\mathbb{J}_T(u)) \cdot \mathbb{M}_T(p)$$

and

(8.3-15) $$\mathbb{R}_T(x,r) = F(\mathbb{H}_T(x)) \cdot \mathbb{N}_T(r)$$

with

(8.3-16) $$\begin{aligned} \Phi_T(x^*) &= G(\mathbb{J}_T(u)) \\ \mathbb{F}_T(u^*) &= F(\mathbb{H}_T(x)) . \end{aligned}$$

Then

$$\pi_T(p^1,p^0 \mid u) = \frac{\mathbb{M}_T(p^1)}{\mathbb{M}_T(p^0)} \tag{8.3-17}$$

$$I_T(x^1,x^0 \mid r) = \frac{\mathbb{H}_T(x^1)}{\mathbb{H}_T(x^0)} \tag{8.3-18}$$

$$\rho_T(r^1,r^0 \mid x) = \frac{\mathbb{N}_T(r^1)}{\mathbb{N}_T(r^0)} \tag{8.3-19}$$

$$0_T(u^1,u^0 \mid p) = \frac{\mathbb{J}_T(u^1)}{\mathbb{J}_T(u^0)}, \tag{8.3-20}$$

where

$$\begin{aligned} \mathbb{H}_T(x) &:= \mathbb{H}(x \cdot [0,T]) \\ \mathbb{J}_T(u) &:= \mathbb{J}(u \cdot [0,T]) . \end{aligned} \tag{8.3-21}$$

The index functions so constructed will not satisfy the test (8.3-13), and the quantity index functions (8.3-18), (8.3-20) are not homogeneous in x^1 and u^1 respectively. Because of these difficulties a different approach may be used for the case where both $x \to \mathbb{P}(x)$ and the inverse correspondence $u \to \mathbb{L}(u)$ are homothetic but may or may not be inversely related in homotheticity.

Recall from the weak duality Proposition (7.1-2) that for $u \to \mathbb{L}(u)$ under weak axioms

$$\mathbb{K}(u,p) = \underset{x}{\text{Min}} \left\{ \langle p,x \rangle : \underline{\bar{\Psi}}(u,x) \geqq 1 \ , \ x \ \varepsilon \ (L_\infty)^n_+ \right\}$$

$$\underline{\bar{\Psi}}(u,x) \leqq \underset{p}{\text{Inf}} \left\{ \langle p,x \rangle : \ \mathbb{K}(u,p) \geqq 1 \ , \ p \ \varepsilon \ (L_1)^n_+ \right\} ,$$

with the equality sign holding for the second expression when the correspondence $u \to \mathbb{L}(u)$ has convex map sets $\mathbb{L}(u)$, axiom $\mathbb{L}$.3SS applies and a weak* topology for $u \to \mathbb{L}(u)$ is used. Because of the possibility of a duality gap, consider

$$(8.3\text{-}22) \quad \underline{\hat{\bar{\Psi}}}_T(u,x) := \underset{p}{\text{Inf}} \left\{ \langle p,x \rangle : \ \mathbb{K}(u,p) \geqq 1 \ , \ p \ \varepsilon \ (L_1)^n_+ \cdot [0,T] \right\} .$$

Then, by symmetry with the definition (8.1-1), (8.1-3) for comparing two vectors of factor price histories, one may define alternatively, for two vectors of factor input rate histories, that

$$(8.3\text{-}23) \quad \begin{aligned} &I : (x^1,x^0) \ \varepsilon \ (L_\infty)^n_+ \times (L_\infty)^n_+ \to I(x^1,x^0 \mid u) \ \varepsilon \ (L_\infty)_+ \ , \\ &I(x^1,x^0 \mid u,T) := \frac{\underline{\hat{\bar{\Psi}}}_T(u,x^1)}{\underline{\bar{\Psi}}_T(u,x^0)} \ , \ T \ \varepsilon \ (0,+\infty) \ . \end{aligned}$$

Similarly, in using the weak duality Proposition (7.2.1), where

$$(8.3\text{-}24) \quad \Omega(x,u) \leqq \underset{r}{\text{Sup}} \left\{ \langle r,u \rangle : \ \mathbb{R}(x,r) \leqq 1 \ , \ r \ \varepsilon \ (L_1)^m \cdot [0,1] \right\} =: \hat{\Omega}(x,u) \ ,$$

an index function for comparing two vectors of output rate histories may be defined by

$$0 : (u^1, u^0) \in (L_\infty)^m_+ \times (L_\infty)^m_+ \to 0(u^1, u^0 \mid x) \in (L_\infty)_+$$

(8.3-25)
$$0(u^1, u^0 \mid x, T) := \frac{\hat{\Omega}_T(x, u^1)}{\hat{\Omega}_T(x, u^0)}, \ T \in (0, +\infty) .$$

In the case of inversely related homothetic correspondences treated previously

(8.3-26)
$$\hat{\underline{\Psi}}_T(u, x) = \frac{\Phi_T(x)}{F^{-1}(\mathbb{F}_T(u))}, \ \hat{\Omega}_T(x, u) = \frac{\hat{\mathbb{F}}_T(u)}{F(\Phi_T(x))},$$

where

(8.3-27)
$$\hat{\Phi}_T(x) = \operatorname*{Inf}_p \left\{ \langle p, x \rangle : \mathbb{M}_T(p) \geqq 1, \ p \in (L_1)^n_+ \cdot [0, T] \right\}$$
$$\hat{\mathbb{F}}_T(u) = \operatorname*{Sup}_r \left\{ \langle r, u \rangle : \mathbb{N}_T(r) \leqq 1, \ r \in (L_1)^m \cdot [0, T] \right\},$$

and

(8.3-28)
$$I_T(x^1, x^0 \mid u) = \frac{\hat{\Phi}_T(x^1)}{\hat{\Phi}_T(x^0)}$$

(8.3-29)
$$0_T(u^1, u^0 \mid x) = \frac{\hat{\mathbb{F}}_T(u^1)}{\hat{\mathbb{F}}_T(u^0)} .$$

It is of interest to compare $\hat{\Phi}_T(x)$ with $\Phi_T(x)$ and $\hat{\mathbb{F}}_T(u)$ with $\mathbb{F}_T(u)$. Recall from (8.3-2) that for the fixed set

$$\mathbb{L}_{\Phi}(1) = \left\{ x \in (L_\infty)^n_+ : \Phi(x) \geqq 1 \right\}$$

the weak duality theorem implies for

$$\mathbb{M}_T(p) = \underset{x}{\mathrm{Min}}\left\{\langle p,x \rangle : \Phi(x) \geqq 1 \ , \ x \ \varepsilon \ (L_\infty)^n_+ \cdot [0,T]\right\}$$

that

$$\Phi_T(x) \leqq \underset{p}{\mathrm{Inf}}\left\{\langle p,x\rangle \ : \ \mathbb{M}_T(p) \geqq 1 \ , \ p \ \varepsilon \ (L_1)^n_+ \cdot [0,T]\right\} = \hat{\Phi}_T(x) \ .$$

If the duality gap is positive, the definitions (8.3-23) and (8.3-25) will not satisfy the test (8.3-13). However, if the weak* topology is used for $(L_\infty)^n_+$ and the correspondence $u \to \mathbb{L}(u)$, has convex map sets and satisfies the strong axiom $\mathbb{L}$.3SS of free disposal of inputs, the test is satisfied and the index function for comparing two vectors of input rate histories is exactly (8.3-10) obtained earlier without the stronger assumptions on the correspondence $u \to \mathbb{L}(u)$. Similarly, under these strong conditions, $\hat{\mathbb{F}}_T(u) = \mathbb{F}_T(u)$ and the alternative definition becomes exactly (8.3-12) obtained earlier without the stronger assumptions. Nothing seems to have been gained here.

On the other hand, if the comparison of minimal cost (and maximal revenue) with product of price index and quantity index is restricted to cases where output (input) varies only by a scale factor, the alternative definition can serve to extend the basis for defining index functions. For this purpose the parent correspondences are assumed to be only ray homothetic (see (5.3-1), (5.3-2)), i.e.,

$$\mathbb{L}(u) = \frac{G(\mathbb{J}(u))}{G\left(\mathbb{J}\left(\frac{u}{\|u\|}\right)\right)} \cdot \mathbb{L}\left(\frac{u}{\|u\|}\right)$$

$$\mathbb{P}(x) = \frac{F(\mathbb{H}(x))}{F\left(\mathbb{H}\left(\frac{x}{\|x\|}\right)\right)} \cdot \mathbb{P}\left(\frac{x}{\|x\|}\right) ,$$

which also covers the case where the correspondences are globally homothetic, but not inversely related in homotheticity. The minimal cost, maximal revenue and distance functionals then take the forms

$$\mathbb{K}_T(u,p) = \frac{G(\mathbb{J}_T(u))}{G\left(\mathbb{J}_T\left(\frac{u}{\|u\|}\right)\right)} \cdot \mathbb{K}_T\left(\frac{u}{\|u\|}, p\right)$$

$$\mathbb{R}_T(x,r) = \frac{F(\mathbb{H}_T(x))}{F\left(\mathbb{H}_T\left(\frac{x}{\|x\|}\right)\right)} \cdot \mathbb{R}_T\left(\frac{x}{\|x\|}, r\right)$$

(8.3-30) $$\bar{\underline{\Psi}}_T(u,x) = \frac{G\left(\mathbb{J}_T\left(\frac{u}{\|u\|}\right)\right)}{G(\mathbb{J}_T(u))} \cdot \bar{\underline{\Psi}}_T\left(\frac{u}{\|u\|}, x\right)$$

(8.3-31) $$\Omega_T(x,u) = \frac{F\left(\mathbb{H}_T\left(\frac{x}{\|x\|}\right)\right)}{F(\mathbb{H}_T(x))} \cdot \Omega_T\left(\frac{x}{\|x\|}, u\right) .$$

Replace $\hat{\bar{\underline{\Psi}}}_T(u,x)$ in (8.2-23) and $\hat{\Omega}_T(x,u)$ in (8.2-25) by $\bar{\underline{\Psi}}(u,x)$ and $\Omega_T(x,u)$ respectively for the alternative definition of the quantity index functions. Then in the case of ray homothetic correspondences

(8.3-32)
$$I : (x^1,x^0) \in (L_\infty)^n_+ \times (L_\infty)^n_+ \to I(x^1,x^0 \mid u) \in (L_\infty)_+$$

$$I_T(x^1,x^0 \mid u) = \frac{\bar{\underline{\Psi}}_T\left(\frac{u}{\|u\|}, x^1\right)}{\bar{\underline{\Psi}}_T\left(\frac{u}{\|u\|}, x^0\right)} , \ T \in (0,+\infty) .$$

$$O : (u^1,u^0) \in (L_\infty)^m_+ \times (L_\infty)^m_+ \to O(u^1,u^0 \mid x) \in (L_\infty)_+$$

(8.3-33)
$$O_T(u^1,u^0 \mid x) = \frac{\Omega_T\left(\frac{x}{||x||}, u^1\right)}{\Omega_T\left(\frac{x}{||x||}, u^0\right)}, \; T \in (0,+\infty) \; .$$

All tests except (8.3-13) are evidently satisfied by (8.3-32) and (8.3-33). For the exceptional one, let (u^1,p^1) and (u^0,p^0) be two output-factor price vectors for comparison of minimal costs. Then

$$\frac{\mathbb{K}_T(u^1,p^1)}{\mathbb{K}_T(u^0,p^0)} = \frac{G\left(\mathbb{J}_T(u^1)\right)}{G\left(\mathbb{J}_T\left(\frac{u^1}{||u^1||}\right)\right)} \cdot \frac{G\left(\mathbb{J}_T\left(\frac{u^0}{||u^0||}\right)\right)}{G\left(\mathbb{J}_T(u^0)\right)} \cdot \frac{\mathbb{K}_T\left(\frac{u^1}{||u^1||}, p^1\right)}{\mathbb{K}_T\left(\frac{u^0}{||u^0||}, p^0\right)} \, ,$$

and using (8.3-30),

(8.3-34)
$$\frac{\mathbb{K}_T(u^1,p^1)}{\mathbb{K}_T(u^0,p^0)} = \frac{\mathbb{K}_T\left(\frac{u^1}{||u^1||}, p^1\right)}{\mathbb{K}_T\left(\frac{u^0}{||u^0||}, p^0\right)} \cdot \frac{\underline{\bar{\Psi}}_T\left(\frac{u^1}{||u^1||}, (x^1)^*\right)}{\underline{\bar{\Psi}}_T\left(\frac{u^0}{||u^0||}, (x^0)^*\right)}$$

where $(x^1)^*$ and $(x^0)^*$ respectively yield $\mathbb{K}_T(u^1,p^1)$ and $\mathbb{K}_T(u^0,p^0)$, for which $\underline{\bar{\Psi}}_T(u^1,(x^1)^*) = \underline{\bar{\Psi}}_T(u^0,(x^0)^*) = 1$. Now, if one compares minimal costs for vectors of output histories along the same ray in $(L_\infty)^m_+$, i.e., if $u^1 = \theta u^0$, $\theta \in (0,+\infty)$, then clearly (8.3-13) is satisfied.

Thus, under the weakest assumptions, i.e., ray homotheticity of input and output structure and not assuming convexity of map sets or free disposability of inputs and outputs, the definition of quantity index functions by ratios of distance functionals weakly dual to the minimal cost and maximal revenue functionals, satisfies the test (8.3-13),

sometimes called Factor Reversal Test, if the reference vectors have the same mix of input (or output) histories.

8.4 Macroeconomic Relationships

For this discussion we shall assume initially that the inversely related dynamic production correspondences are inversely related globally homothetic.

Let $\mathbb{K}_T$ and $\mathbb{R}_T$ denote the values of $\mathbb{K}_T(u,p)$ and $\mathbb{R}_T(x,r)$ respectively. From (8.3-1) and (8.3-4)

$$\mathbb{K}_T = \bar{F}^{1}(\mathrm{ff}_T(u)) \cdot \mathbb{M}_T(p) \ , \ T \in [0,+\infty) \tag{8.4-1}$$

$$\mathbb{R}_T = F(\phi\!\!\!\phi_T(x)) \cdot \mathbb{N}_T(r) \ , \ T \in [0,+\infty) \ . \tag{8.4-2}$$

As discussed above, $\mathbb{M}_T(p)$ and $\mathbb{N}_T(r)$ are price level functionals, $T \in (0,+\infty)$. Denote the values of those two "price levels" by $\mathbb{M}_T$ and $\mathbb{N}_T$ respectively. From (8.3-7) and (8.3-8), it follows that

$$\mathbb{K}_T = \phi\!\!\!\phi^*_T \cdot \mathbb{M}_T \ , \ T \in [0,+\infty) \tag{8.4-3}$$

$$\mathbb{R}_T = \mathrm{ff}^*_T \cdot \mathbb{N}_T \ , \ T \in [0,+\infty) \tag{8.4-4}$$

where $\phi\!\!\!\phi^*_T$ and ff^*_T are the values of the functionals $\phi\!\!\!\phi_T(x)$ and $\mathrm{ff}_T(u)$ for x^* yielding $\mathbb{K}_T$ and u^* yielding $\mathbb{R}_T$. Now as shown by the previous formulations of index functions, $\phi\!\!\!\phi_T(x)$ and $\mathrm{ff}_T(u)$ are index functions of input level and output level respectively (see (8.3-10) and (8.3-12)). Thus (8.4-3) and (8.4-4) are valid macroeconomic statements: that minimal cost and maximal revenue over $[0,T]$ are the product of optimal factor (output) level and factor (output)

price level over $[0,T]$ yielding the minimal (maximal) value.

With the same definition of macroeconomic variables, (8.4-1) and (8.4-2) yield

$$\mathbb{ff}_T = F\left(\frac{\mathbb{K}_T}{\mathbb{M}_T}\right), \; T \in [0,+\infty) \tag{8.4-5}$$

$$\mathbb{\phi}_T = F^{-1}\left(\frac{\mathbb{R}_T}{\mathbb{N}_T}\right), \; T \in [0,+\infty) \; . \tag{8.4-6}$$

The macroeconomic statement of (8.4-5) is that the "level" of output histories over $[0,T]$ is a function F of the factor price level deflated minimal cost available over $[0,T]$ to support the output level, i.e., it is a macroeconomic indirect production function in real terms. Similarly, (8.4-6) states that the "level" of input histories over $[0,T]$ required to attain a price deflated revenue over $[0,T]$ is a function F^{-1} of this real revenue.

From (8.3-7) and (8.3-8) one obtains the following two macroeconomic production functions

$$\mathbb{ff}^*_T = F(\mathbb{\phi}_T) \; , \; T \in [0,+\infty) \tag{8.4-7}$$

$$\mathbb{\phi}^*_T = F^{-1}(\mathbb{ff}_T) \; , \; T \in [0,+\infty) \; . \tag{8.4-8}$$

Equation (8.4-7) states that the level of revenue maximizing output histories over $[0,T]$ is a function of the level of input histories available over $[0,T]$. Inversely, (8.4-8) states that the level of minimal cost input histories over $[0,T]$ to attain a given level $\mathbb{ff}_T$ of output histories over $[0,T]$ is a function F^{-1} of the latter.

If the system is operated at minimal cost and maximal revenue, the macroeconomic production functions (8.4-5), (8.4-7) and (8.4-6), (8.4-7) may be combined to obtain

$$\frac{\mathbb{R}_T}{\mathbb{N}_T} = F\left(\frac{\mathbb{K}_T}{\mathbb{M}_T}\right),$$

i.e., over $[0,T]$ real maximal revenue is a function F of real minimal cost, an interesting relationship for estimating the function F which plays the role of defining returns to scale in macroeconomic terms. Of course we have assumed:

(a) Inversely related globally homothetic production correspondences

(b) Operation simultaneously at minimal cost and maximal revenue.

Also the definitions of the function $\mathbb{M}_T$, $\mathbb{N}_T$, $\mathbb{F}_T$, Φ_T , $T \in [0,+\infty)$ and the relationships between them must be observed in calculating the prive level deflators $\mathbb{M}_T$ and $\mathbb{N}_T$, $T \in [0,+\infty)$.

CHAPTER 9

INDIRECT DYNAMIC PRODUCTION CORRESPONDENCES†

In the preceding chapters production structure has been defined in purely physical terms by correspondences relating vectors of input rate histories to subsets of vectors of output rate histories obtainable thereby, or inversely by relating vectors of output rate histories to subsets of vectors of input rate histories yielding at least these output rate histories. However, indirectly for some production decisions, particularly those involving a comparison of cost and return to be obtained from production (as in cost-benefit analysis), one would like to relate cost to subsets of output histories obtainable with this monetary resource, or to relate return to subsets of input rate histories yielding at least that return. Here, vectors of factor price histories in the first case, and vectors of output price histories are involved in defining the correspondences.

9.1 Cost-Indirect Output Correspondence

Consider a vector $p \varepsilon (L_1)^n_+$ of factor price histories. For any vector $u \varepsilon (L_\infty)^m_+$ of output rate histories, the minimal cost of obtaining u under the price structure p is defined in terms of the input correspondence $u \to \mathbb{L}(u)$ by the cost functional $\mathbb{K}(u,p)$. Let c be any positive monetary value. The Cost Indirect Output Correspondence is a mapping

†The contents of this chapter are a revision of Shephard (1977) and extension of Shephard (1974a).

$$(9.1\text{-}1) \qquad \begin{pmatrix}\underline{p}\\ c\end{pmatrix} \varepsilon\ (L_1)^n_+ \rightarrow \mathbb{\Gamma}\begin{pmatrix}\underline{p}\\ c\end{pmatrix} \varepsilon\ 2^{(L_\infty)^m_+}$$

where

$$(9.1\text{-}2) \qquad \mathbb{\Gamma}\begin{pmatrix}\underline{p}\\ c\end{pmatrix} : = \left\{u\ \varepsilon\ (L_\infty)^m_+ : \mathbb{K}(u,p) \leqq c\right\}$$

is the set of vectors of output rate histories obtainable at total cost not exceeding c, under the price structure p. Equivalently, $\mathbb{\Gamma}\begin{pmatrix}\underline{p}\\ c\end{pmatrix}$ may be defined by

$$(9.1\text{-}3) \qquad \mathbb{\Gamma}\begin{pmatrix}\underline{p}\\ c\end{pmatrix} : = \left\{\cup\ \mathbb{P}(x) : x\ \varepsilon\ (L_\infty)^n_+ , \langle p,x\rangle \leqq c\right\} .$$

To see this note:

(a) If $u\ \varepsilon\ \mathbb{\Gamma}\begin{pmatrix}\underline{p}\\ c\end{pmatrix}$, there exists $x^*\ \varepsilon\ (L_\infty)^n_+$ such that $\mathbb{K}(u,p) = \langle p,x^*\rangle \leqq c$, implying that $u\ \varepsilon \cup \mathbb{P}(x)$ for $\langle p,x\rangle \leqq c$.

(b) Conversely, if $u\ \varepsilon \cup \mathbb{P}(x)$ for $\langle p,x\rangle \leqq c$, then for some $x^o\ \varepsilon\ (L_\infty)^n_+ : \mathbb{K}(u,p) \leqq \langle p,x^o\rangle \leqq c$ and $u\ \varepsilon \left\{u\ \varepsilon\ (L_\infty)^m_+ : \mathbb{K}(u,p) \leqq c\right\}$.

The definition of the Cost Indirect Output Correspondence expressed by (9.1-2) does not guarantee that the set $\mathbb{\Gamma}(p/c)$ is compact or even bounded. There are several possibilities for $\mathbb{\Gamma}(p/c)$ to be compact. First, as a practical matter one would not expect that input rate histories unbounded in the Ess Sup norm to be of interest. Hence, without serious loss of generality, the allowable vectors $x\ \varepsilon\ (L_\infty)^n_+$ may be restricted to a subset $\nabla \subset (L_\infty)^n_+$ defined by

$$(9.1\text{-}4) \quad \nabla : = \left\{x\ \varepsilon\ (L_\infty)^n_+ : ||x_i|| \leqq A_i\ ,\ A_i \gg 0\ ,\ i\ \varepsilon\ \{1,2,\ \ldots,\ n\}\right\} .$$

Then $\mathbb{T}\left(\frac{p}{c}\right) \subset \bigcup_{x\varepsilon\nabla} \mathbb{P}(x)$, since

$$\{x \in \nabla : \langle p,x\rangle \leqq c\} \subset \nabla .$$

Now $\bigcup_{x\varepsilon\nabla} \mathbb{P}(x)$ is bounded, because if there exists an infinite sequence $\{u^{\alpha}\} \subset \bigcup_{x\varepsilon\nabla} \mathbb{P}(x)$ of output vectors with $\{||u^{\alpha}||\} \to +\infty$, there exists corresponding to $\{u^{\alpha}\}$ an infinite sequence $\{x^{\alpha}\}$ with $u^{\alpha} \in \mathbb{P}(x^{\alpha})$ such that $\{||x^{\alpha}||\} \to +\infty$, contradicting (9.1-4). Alternatively, if Property $\mathbb{P}$.3SS holds and $x \in \nabla$, then $x \leq y$ where $y_i = \operatorname*{Sup}_{\nabla} ||x_i||$. Since $||y||$ is bounded and $\mathbb{P}(x) \subset \mathbb{P}(y)$ for $x \in \nabla$, it follows that $\bigcup_{x\varepsilon\nabla} \mathbb{P}(x)$ is bounded. Accordingly, under a weak* topology for output vectors,

$$\mathbb{T}\left(\frac{p}{c} \mid \nabla\right) = \left\{u \in \cup\ \mathbb{P}(x) : x \in \nabla , \langle p,x\rangle \leqq c\right\}$$

is relatively compact. Further if Property $\mathbb{P}$.5 is taken as holding under the weak* topology, $\mathbb{T}\left(\frac{p}{c} \mid \nabla\right)$ is compact.

On the other hand, if price vectors for inputs are such that $p \ggg 0$, i.e., price histories are strictly positive except on subsets of $[0,+\infty)$ of measure zero, the set

$$S_c := \left\{x \in (L_\infty)^n_+ : \langle p,x\rangle \leqq c\right\}$$

of admissable input vectors x is bounded. Then boundedness and compactness of $\mathbb{T}(p/c)$ follow as described when vectors of input rate histories were restricted to ∇ .

For either of the two cases described above, i.e., $x \in \nabla$ or $p \ggg 0$, the set S_c is relatively compact under a weak* topology for vectors x of input rate histories. Then if Property $\mathbb{P}$.2S is applied

with $\mathbb{P}.5$ and $\mathbb{P}.5+$ (see Section 2.1) $\mathbb{I}\left(\frac{p}{c}\right)$ is compact, because the following proposition may be applied.

Proposition (9.1-1):

Property $\mathbb{P}.5+$ holds for the correspondence $x \in (L_\infty)_+^n \rightarrow \mathbb{P}(x) \in 2^{(L_\infty)_+^m}$ with $\mathbb{P}.2S$ and $\mathbb{P}.5$ holding (i.e., $x \rightarrow \mathbb{P}(x)$ is compact valued), iff $\bigcup_{x \in Y} \mathbb{P}(x)$ is compact when Y is a compact subset of $(L_\infty)_+^n$.

Proposition (9.1-1) is a paraphrasing of the characterization of upper semi-continuity by sequences. See Hildenbrand (1974). Thus with the stronger properties $\mathbb{P}.5+$ and $\mathbb{P}.2S$ with $\mathbb{P}.5$ under the norm topology for vectors of output histories, compact cost indirect outputs sets $\mathbb{I}\left(\frac{p}{c}\right)$ may be obtained under a weak* topology for vectors of input rate histories when x is restricted to ∇ or $p >>> 0$.

Not all price histories p_i need be strictly positive for bounded $\mathbb{I}(p/c)$, if there exists an essential proper subset of factors which is strong limitational. Strictly positive price histories for such factors implies that the input rate histories of these factors are bounded, and then it follows that $\mathbb{P}(x)$ is bounded for any $x \in S_c$, (see Section 3.3) even though S_c is not a bounded subset of $(L_\infty)_+^m$.

For easy reference the properties of the cost-indirect output correspondence are stated in the following proposition.

Proposition (9.1-2):

The properties of the cost-indirect output correspondence

$\left(\frac{p}{c}\right) \in (L_1)_+^n \rightarrow \mathbb{I}\left(\frac{p}{c}\right) \in 2^{(L_\infty)_+^m}$ are:

$\mathbb{\Gamma}.1$ $\quad \mathbb{\Gamma}(0) = (L_\infty)^m_+$ and $0 \in \mathbb{\Gamma}\left(\frac{p}{c}\right)$ for all $p \in (L_1)^n_+$, $c > 0$.

$\mathbb{\Gamma}(0 \mid A) = \bigcup_{x \in A} \mathbb{P}(x)$, where A is ∇ or S_c .

$\mathbb{\Gamma}.2$ $\quad \mathbb{\Gamma}\left(\frac{p}{c}\right)$ is bounded if (a) $p \in (L_1)^n_+$, $c > 0$, $x \in (L_\infty)^n_+ \cap \nabla$, or (b) $p >\!>\!> 0$, or (c) price histories are strictly positive for a strongly limitational subset of factors.

$\mathbb{\Gamma}.2S$ $\quad \mathbb{\Gamma}\left(\frac{p}{c}\right)$ is totally bounded (relatively compact) if:

(i) (a), (b) or (c) of Property $\mathbb{\Gamma}.2$ apply with a weak* topology for vectors u of output rate histories, or

(ii) the norm topology is used for output vectors and a weak* topology is used for input vectors with $\mathbb{P}.2S$, and $\mathbb{P}.5$ with $\mathbb{P}.5+$ is invoked for the parent correspondence $x \to \mathbb{P}(x)$.

$\mathbb{\Gamma}.3$ $\quad \mathbb{\Gamma}\left(\left(\frac{p}{c}\right)'\right) \supset \mathbb{\Gamma}\left(\left(\frac{p}{c}\right)\right)$ for $\left(\frac{p}{c}\right)' \leqq \left(\frac{p}{c}\right)$.

$\mathbb{\Gamma}.4$ $\quad$ If $\mathbb{L}(u) \neq \emptyset$, $p \in (L_1)^n_+$, $c > 0$, there exists a positive scalar θ such that $u \in \mathbb{\Gamma}\left(\theta\, \frac{p}{c}\right)$.

$\mathbb{\Gamma}.5$ $\quad$ If $\mathbb{K}(u,p)$ is lower semi-continuous in $u \in (L_\infty)^m_+$, $\mathbb{\Gamma}\left(\frac{p}{c}\right)$ is closed.

$\mathbb{\Gamma}.6$ $\quad$ If $u \in \mathbb{\Gamma}\left(\frac{p}{c}\right)$, $(\theta u) \in \mathbb{\Gamma}\left(\frac{p}{c}\right)$ for $\theta \in [0,1]$.

$\mathbb{\Gamma}.6S$ $\quad$ If $u \in \mathbb{\Gamma}\left(\frac{p}{c}\right)$, $(\theta_1 u_1, \theta_2 u_2, \ldots, \theta_m u_m) \in \mathbb{\Gamma}\left(\frac{p}{c}\right)$ for $\theta_i \in [0,1]$, $i \in \{1,2,\ldots,m\}$.

$\mathbb{\Gamma}.6SS$ $\quad$ If $u \in \mathbb{\Gamma}\left(\frac{p}{c}\right)$, $\{v \in (L_\infty)^m_+ : 0 \leqq v \leqq u\} \subset \mathbb{\Gamma}\left(\frac{p}{c}\right)$.

Certain similarities may be observed between the map sets $\mathbb{\Gamma}\left(\frac{p}{c}\right)$ of the cost-indirect output correspondence and the map sets $\mathbb{P}(x)$ of the parent dynamic production correspondence. The null vector of output rate histories belongs to both, and the disposable structure of the ouput

set $\Gamma\left(\frac{p}{c}\right)$ parallels that of $\mathbb{P}(x)$. The boundedness and relative compactness of $\Gamma\left(\frac{p}{c}\right)$ are similar to that of $\mathbb{P}(x)$, with some complication. $\Gamma\left(\frac{p}{c}\right)$ is monotone nonincreasing in $\left(\frac{p}{c}\right)$ while $\mathbb{P}(x)$ has a comparable nondecreasing relation to x only if Property $\mathbb{P}$.3SS holds. Every feasible vector u of output rate histories, i.e., $\mathbb{L}(u) \neq \emptyset$, may be attained in a set $\Gamma\left(\theta_u \frac{p}{c}\right)$ by sufficiently small positive scalar θ_u , but no comparable global property holds for the parent correspondence $x \to \mathbb{P}(x)$.

9.2 Return-Indirect Input Correspondence

Consider a vector $r \in (L_1)^m$ of output price histories. For any vector $x \in (L_\infty)_+^m$ of input rate histories, the maximal revenue obtainable under the price structure r is defined in terms of the output correspondence $x \to \mathbb{P}(x)$ by the return functional $\mathbb{R}(x,r)$. Let $R > 0$ be any positive value. The return-indirect input correspondence is defined by

$$(9.2\text{-}1) \qquad \left(\frac{r}{R}\right) \in (L_1)^m \to \Pi\left(\frac{r}{R}\right) \in 2^{(L_\infty)_+^n} , \; R \in (0,+\infty)$$

$$(9.2\text{-}2) \qquad \Pi\left(\frac{r}{R}\right) := \left\{x \in (L_\infty)_+^n : \mathbb{R}(x,r) \geqq R\right\} ,$$

where $\Pi\left(\frac{r}{R}\right)$ is the set of vectors of input rate histories yielding at least the total return R under the price structure r . An equivalent definition of $\Pi\left(\frac{r}{R}\right)$ is

$$(9.2\text{-}3) \qquad \Pi\left(\frac{r}{R}\right) := \left\{\cup \; \mathbb{L}(u) : u \in (L_\infty)_+^m \, . \, \langle r,u\rangle \geqq R\right\} .$$

Verification of (9.2-3) is analogous to that made for (9.1-2).

Clearly $\mathbb{L}\left(\frac{r}{R}\right)$ is empty for $r \leqq 0$, and $0 \notin \mathbb{L}\left(\frac{r}{R}\right)$, $r > 0$.

Consider $\bigcap_{\alpha=1}^{\infty} \mathbb{L}\left(\frac{r}{R_\alpha}\right)$, for $\{R_\alpha\} \to \infty$, where $\mathbb{R}(x,r) > 0$ for $\mathbb{P}(x) \neq \{0\}$. Assume $x \in \mathbb{L}\left(\frac{r}{R_\alpha}\right)$, $\alpha = 1,2, \ldots,$. Since $R_\alpha \to \infty$ and $x \in \mathbb{L}(u^\alpha)$ for $\langle r,u^\alpha\rangle \geqq R_\alpha$ by (9.2-3), it follows that $x \in \bigcap_{\alpha=1}^{\infty} \mathbb{L}(u^\alpha)$ for $\{||u^\alpha||\} \to +\infty$, contradicting Property $\mathbb{L}$.2. Thus $\bigcap_{\alpha=1}^{\infty} \mathbb{L}\left(\frac{r}{R_\alpha}\right)$ is empty for $r \in (L_1)_+^m$ such that $\mathbb{R}(x,r) > 0$ for $\mathbb{P}(x) \neq \{0\}$, and $\{R_\alpha\} \to +\infty$.

Concerning disposability of input histories for vectors $x \in \mathbb{L}\left(\frac{r}{R}\right)$, $(\lambda x) \in \mathbb{L}\left(\frac{r}{R}\right)$ for $\lambda \in [1,+\infty)$ because $x \in \mathbb{L}\left(\frac{r}{R}\right)$ implies $\mathbb{R}(x,r) \geqq R$ and $\mathbb{R}(\lambda x,r) \geqq R$ for $\lambda \in [1,+\infty)$ by Property $\mathbb{R}$.8 for the return functional when outputs are weakly disposable (see Proposition (6.2-1)). In case Property $\mathbb{P}$.6S or Property $\mathbb{P}$.6SS applies for the parent correspondence $x \to \mathbb{P}(x)$, $(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n) \in \mathbb{L}\left(\frac{r}{R}\right)$ for $\lambda_i \in [1,+\infty)$, $i \in \{1,2, \ldots, n\}$ or $x' \in \mathbb{L}\left(\frac{r}{R}\right)$ for $x' \geqq x$.

Next suppose $x \in (L_\infty)_+^n$ such that $\mathbb{P}(x) \neq \{0\}$ and $r \in (L_1)^m$ such that $\mathbb{R}(x,r) > 0$. Then due to Property $\mathbb{R}$.3 for the return functional, there exists a positive scalar λ such that for $\mathbb{R} > 0$, $\mathbb{R}(x,\lambda r) \geqq R$, i.e., $x \in \mathbb{L}\left(\lambda \frac{r}{R}\right)$.

If the return functional $\mathbb{R}(x,r)$ is upper semi-continuous in $x \in (L_\infty)_+^n$, $\mathbb{L}\left(\frac{r}{R}\right)$ is closed.

Finally, the correspondence $\left(\frac{r}{R}\right) \to \mathbb{L}\left(\frac{r}{R}\right)$ is strongly monotone increasing in $\left(\frac{r}{R}\right)$ due to Property $\mathbb{R}$.5 for the functional $\mathbb{R}(x,r)$.

The following proposition summarizes these facts.

Proposition (9.2-1):

The properties of the return-indirect production correspondence

$\left(\frac{r}{R}\right) \varepsilon\, (L_1)^m \rightarrow \Pi\left(\frac{r}{R}\right) \varepsilon\, 2^{(L_\infty)^n_+}$, $R > 0$, are

II.1 $\Pi\left(\frac{r}{R}\right)$ is empty for $r \leqq 0$ and $0 \notin \Pi\left(\frac{r}{R}\right)$.

II.2 $\bigcap_{\alpha=1}^{\infty} \Pi\left(\frac{r}{R_\alpha}\right)$ is empty for $r \;\varepsilon\; (L_1)^m_+$ such that $\mathbb{R}(x,r) > 0$ for $\mathbb{P}(x) \neq \{0\}$, when $R_\alpha \rightarrow +\infty$.

II.3 If $x \;\varepsilon\; \Pi\left(\frac{r}{R}\right)$, $(\lambda x) \;\varepsilon\; \Pi\left(\frac{r}{R}\right)$ for $\lambda \;\varepsilon\; [1,+\infty)$.

II.3S If $x \;\varepsilon\; \Pi\left(\frac{r}{R}\right)$, $(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n) \;\varepsilon\; \Pi\left(\frac{r}{R}\right)$ for $\lambda_i \;\varepsilon\; [1,+\infty)$, $i \;\varepsilon\; \{1,2, \ldots, n\}$.

II.3SS If $x \;\varepsilon\; \Pi\left(\frac{r}{R}\right)$, $x' \;\varepsilon\; \Pi\left(\frac{r}{R}\right)$ for $x' \geqq x$.

II.4 For $x \;\varepsilon\; (L_\infty)^n_+$ with $\mathbb{P}(x) \neq \{0\}$, and $r \;\varepsilon\; (L_1)^m$ with $\mathbb{R}(x,r) > 0$, there exists a positive scalar λ such that $x \;\varepsilon\; \Pi\left(\frac{\lambda r}{R}\right)$.

II.5 If the return functional $\mathbb{R}(x,r)$ is upper semi-continuous in x , $\Pi\left(\frac{r}{R}\right)$ is closed.

II.6 $\Pi\left(\left(\frac{r}{R}\right)'\right) \supset \Pi\left(\frac{r}{R}\right)$ for $\left(\frac{r}{R}\right)' \geqq \left(\frac{r}{R}\right)$.

Here too one observes similarities between the indirect and direct production correspondence. For $r \leqq 0$, $\Pi\left(\frac{r}{R}\right)$ is empty, whereas $\mathbb{L}(0) = (L_\infty)^n_+$, and similar to the property for the correspondence $u \rightarrow \mathbb{L}(u)$, the null vector does not belong to the set $\Pi\left(\frac{r}{R}\right)$ of vectors of input histories when it is nonempty. For $\{R_\alpha\} \rightarrow +\infty$, the intersection of the sets $\Pi\left(\frac{r}{R_\alpha}\right)$ is empty, like Property L.2. The disposal properties II.3, II.3S, II.3SS are the same as those for the map sets $\mathbb{L}(u)$, but

Property $\mathbb{L}.6$ takes the strong form in all cases with inclusion property which is the reverse of that for the parent correspondence $u \to \mathbb{L}(u)$.

9.3 Cost-Benefit Analysis Under Global Prices

The result of expending a positive total cost c is a set $\mathbb{\Gamma}\left(\frac{p}{c}\right) \subset (L_\infty)^m_+$ of alternative vectors of output rate histories which may be obtained under the price structure $p \in (L_1)^n_+$. Consider the case where a vector $r \in (L_1)^m$ of price histories for outputs may be applied globally to evaluate the vectors of output histories of the set $\mathbb{\Gamma}\left(\frac{p}{c}\right)$. With benefit expressed in these terms, one may relate to each positive cost rate c the functional

$$(9.3\text{-}1) \qquad \mathbb{B}\left(\frac{p}{c},r\right) : = \operatorname*{Sup}_{u} \left\{\langle r,u\rangle : u \in \mathbb{\Gamma}\left(\frac{p}{c}\right)\right\}$$

as the corresponding maximal benefit which may be obtained at total cost not exceeding c with price history structure p .

Under the assumption that global price histories r apply, the functional $\mathbb{B}\left(\frac{p}{c}, r\right)$ establishes a cardinal relationship between total cost c and supremal return obtainable from c , in the sense that $\mathbb{B}\left(\frac{p}{c}, r\right)$ measures the amount of return.

Clearly $\mathbb{B}\left(\frac{p}{c},r\right)$ is zero if $r \leqq 0$. On the other hand, if $r \geq 0$ and $p = 0$, $\mathbb{B}\left(\frac{p}{c},r\right)$ is $+\infty$, and $\mathbb{B}\left(\frac{p}{c},r\right) \geqq 0$ for all $p \in (L_1)^n_+$, $r \in (L_1)^m$. When $p >>> 0$ or $p_i > 0$ for $i \in \{\nu_1,\nu_2, \ldots, \nu_k\}$, where $\{\nu_1,\nu_2, \ldots, \nu_k\}$ is a strongly limitational essential subset of the factors, $\mathbb{B}\left(\frac{p}{c},r\right) < +\infty$. Otherwise, in $r \in (L_1)^m$, the functional $\mathbb{B}\left(\frac{p}{c},r\right)$ has the same properties as the return functional $\mathbb{R}(x,r)$. Due to the strong property $\mathbb{\Gamma}.3$, $\mathbb{B}\left(\frac{p}{c},r\right)$ is nonincreasing in $\left(\frac{p}{c}\right) \in (L_1)^n_+$, whereas $\mathbb{R}(x,r)$ may take one or other of the non-decreasing Properties $\mathbb{R}.8$, $\mathbb{R}.8S$, $\mathbb{R}.8SS$ depending upon whether

$\mathbb{P}.3$, $\mathbb{P}.3S$ or $\mathbb{P}.3SS$ holds for the parent correspondence $x \to \mathbb{P}(x)$.

For given global price histories $p^o \in (L_1)^n_+$, $r^o \in (L_1)^m$, the Benefit Functional $\mathbb{B}\left(\frac{p^o}{c}, r^o\right)$ depends only upon the total cost $c > 0$. The cost-benefit relationship is then expressed as the function:

$$(9.3\text{-}2) \qquad c \in R_+ \to B(c \mid p^o, r^o) := \mathbb{B}\left(\frac{p^o}{c}, r^o\right) \in R_+ .$$

Clearly $B(c \mid p^o, r^o)$ is nondecreasing in c .

A cost-benefit relationship can also be formulated inversely in terms of the minimal cost of achieving a total benefit $B \in R_{++}$ under a price structure r , by the functional

$$(9.3\text{-}3) \qquad \mathbb{X}\left(\frac{r}{B}, p\right) := \operatorname*{Inf}_{x} \left\{\langle p, x\rangle : x \in \Pi\left(\frac{r}{B}\right)\right\}$$

in which $p \in (L_1)^n_+$ is taken as applicable globally. If $r \leqq 0$, $\Pi\left(\frac{r}{B}\right)$ is empty and $\mathbb{X}\left(\frac{r}{B}, p\right) = +\infty$. If $r \nleqq 0$ with $\Pi\left(\frac{r}{B}\right)$ not empty, $\mathbb{X}\left(\frac{r}{B}, p\right) = 0$ for $p = 0$ and $\mathbb{X}\left(\frac{r}{B}, p\right) > 0$ if $p >>> 0$. Otherwise, $\mathbb{X}\left(\frac{r}{B}, p\right)$ has the same properties as the cost functional $\mathbb{K}(u,p)$ in $p \in (L_1)^n_+$. The strong property $\Pi.6$ implies that $\mathbb{X}\left(\frac{r}{B}, p\right)$ is nonincreasing in $\left(\frac{r}{B}\right)$.

Then for given price histories $(p^o) \in (L_1)^n_+$, $(r^o) \in (L_1)^m$, the minimal cost of achieving at least a total benefit B is given by the function

$$(9.3\text{-}4) \qquad B \in R_+ \to \chi(B \mid r^o, p^o) := \mathbb{X}\left(\frac{r^o}{B}, p^o\right) \in R_+ .$$

Since $\chi\!\left(\frac{r}{B}, p\right)$ is nonincreasing in $\left(\frac{r}{B}\right)$, the function $\chi(B \mid r^{o}, p^{o})$ is nondecreasing in B.

In case the parent correspondences $x \to \mathbb{P}(x)$, $u \to \mathbb{L}(u)$ are inversely related homothetic, these cost-benefit relationships take simple interesting forms:

$$\mathbb{P}(x) = F(\Phi(x)) \cdot \left\{u \in (L_\infty)^{m}_{+} : \mathbb{F}(u) \leqq 1\right\}$$

$$\mathbb{L}(u) = F^{-1}(\mathbb{F}(u)) \cdot \left\{x \in (L_\infty)^{n}_{+} : \Phi(x) \geqq 1\right\},$$

where $\mathbb{F}(u)$ and $\Phi(x)$ are homogeneous distance functionals (see Section 5.1). Then the map sets $\Gamma\!\left(\frac{p}{c}\right)$ and $\Pi\!\left(\frac{r}{R}\right)$ are expressed by

$$\Gamma\!\left(\frac{p}{c}\right) = \left\{u \in (L_\infty)^{m}_{+} : \mathbb{F}(u) \leqq F^{-1}\!\left(\frac{c}{\mathbb{M}(p)}\right)\right\} \tag{9.3-5}$$

$$\Pi\!\left(\frac{r}{R}\right) = \left\{x \in (L_\infty)^{n}_{+} : \Phi(x) \geqq F\!\left(\frac{R}{\mathbb{N}(p)}\right)\right\} \tag{9.3-6}$$

where $\mathbb{M}(p)$ and $\mathbb{N}(p)$ are price level deflators (see Chapter 8). The isoquants of $\Gamma\!\left(\frac{p}{c}\right)$ and $\Pi\!\left(\frac{r}{R}\right)$ are then expressed in price level deflated total cost c and total return R by

$$\text{ISOQ } \Gamma\!\left(\frac{p}{c}\right) = \left\{u \in (L_\infty)^{m}_{+} : \mathbb{F}(u) = F^{-1}\!\left(\frac{c}{\mathbb{M}(p)}\right)\right\} \tag{9.3-7}$$

$$\text{ISOQ } \Pi\!\left(\frac{r}{R}\right) = \left\{x \in (L_\infty)^{n}_{+} : \Phi(x) = F\!\left(\frac{R}{\mathbb{N}(r)}\right)\right\}. \tag{9.3-8}$$

In using these special forms one obtains

$$\mathbb{B}\left(\frac{p}{c}, r\right) = \sup_u \left\{ \langle r,u \rangle : \mathbb{F}(u) \leqq F^{-1}\left(\frac{c}{\mathbb{M}(p)}\right) \right\}$$

$$= F^{-1}\left(\frac{c}{\mathbb{M}(p)}\right) \cdot \sup_u \{ \langle r,u \rangle : \mathbb{F}(u) \leqq 1 \} ,$$

and by (8.3-5),

$$\mathbb{B}\left(\frac{p}{c}, r\right) = F^{-1}\left(\frac{c}{\mathbb{M}(p)}\right) \cdot \mathbb{N}(r) . \tag{9.3-10}$$

Similarly

$$\mathbb{X}\left(\frac{r}{B}, p\right) = F\left(\frac{B}{\mathbb{N}(r)}\right) \cdot \mathbb{M}(p) \tag{9.3-11}$$

using (9.3-8) and (8.3-2). Recall that $\mathbb{M}(p)$ and $\mathbb{N}(r)$ are total value price level deflators. In such terms (9.3-10) states that the Maximal Benefit obtainable at a total cost not exceeding $c > 0$, under price histories $p \varepsilon (L_1)^n_+$ and globally effective price histories $r \varepsilon (L_1)^m$, is directly proportional to the price level deflator of total value of outputs, and varies by $F^{-1}(\cdot)$ with total cost deflated value $\frac{c}{\mathbb{M}(p)}$. Inversely, the Minimal Cost of achieving at least a benefit $B > 0$, under global price histories $p \varepsilon (L_1)^n_+$ and price histories $r \varepsilon (L_1)^m$ for outputs, varies directly with the price level deflator of total cost and by $F(\cdot)$ with total benefit deflated value $\frac{B}{\mathbb{N}(r)}$.

To put the interpretation of (9.3-10) and (9.3-11) another way, Equations (9.3-10) and (9.3-11) may be written as

$$\frac{\mathbb{B}\left(\frac{p}{c}, r\right)}{\mathbb{N}(r)} = F^{-1}\left(\frac{c}{\mathbb{M}(p)}\right) \tag{9.3-12}$$

$$(9.3\text{-}13) \qquad \frac{\mathbb{X}\left(\frac{r}{B}, p\right)}{\mathbb{M}(p)} = F\left(\frac{B}{\mathbb{N}(r)}\right)$$

stating respectively that

(a) The Real Maximal Benefit attainable at a total cost not exceeding c is a function $F^{-1}(\cdot)$ of the Real Value of Cost c ,

(b) The Real Minimal Cost of achieving at least a total benefit B is a function $F(\cdot)$ of the Real Value of Benefit B .

Such relationships afford a possibly interesting way of estimating returns to scale when the parent correspondences are inversely related homothetic. Then

$$\mathbb{P}(x) = F(\phi(x)) \cdot \mathbb{P}(1) ,$$

where $\mathbb{P}(1)$ is a fixed set and $\phi(x)$ may be interpreted as an index of level of inputs (see Section 8.3), and the function $F(\cdot)$ defines in this sense the returns to scale of input histories, which may be studied by (9.3-13) as the smoothed graph of Real Cost of Achieving B plotted on the Real Value of B . The price level deflators find their definitions in terms of the distance functionals $\phi(x)$ and $\mathbb{f}(u)$ and the fixed sets $\mathbb{L}(1)$ and $\mathbb{P}(1)$ as minimal and maximal problems. (See Section 8.3).

If the output price histories $r \in (L_1)^m$ are taken as given, and the price histories $p \in (L_1)^n_+$ are known, the maximal benefit $\mathbb{B}\left(\frac{p}{c}, r\right)$ varies directly with real total cost c by the function $F^{-1}(\cdot)$. See (9.3-12). Similarly for given histories of prices p and r , the minimal cost $\mathbb{X}\left(\frac{r}{B}, p\right)$ varies directly with the real total benefit B by the function

$F(\cdot)$, see (9.3-13), indicating how the cost benefit relationship under global prices is related to returns to scale.

9.4 Cost-Benefit Analysis Under A Representative Preference Structure

In the previous section the structure of a cost benefit analysis under global prices reflects benefit to the recipients of outputs only if a fixed set of price histories are compatible with the underlying preferences. To some extent this may be true if, for example, the price histories $r \in (L_1)^m$ evaluating output histories express time preferences, and the price histories $p \in (L_1)^n_+$ reflect actual anticipated costs, as they might if the evaluation is made over a bounded interval of time. However, fixed price histories cannot be made to express changing preferences for different relative amounts of outputs.

In any cost-benefit relationship, the benefit to many individuals may be involved. Thus, even if a preference structure is used to evaluate outputs, there may be no single relationship to consistently express the preferences of all individuals.

For the discussion of this section, a single preference structure (by delegated authority, perhaps) will be assumed in the form of a functional

$$(9.4\text{-}1) \qquad u \in (L_\infty)^m_+ \to \mathbb{PF}(u) \in R^1 .$$

Let $\gtrsim$ denote "at least as preferred as," and $\sim$ denote indifference, i.e., $\gtrsim$ and $\lesssim$. Then the preference functional (9.4-1) obeys for $u \in (L_\infty)^m_+$, $v \in (L_\infty)^m_+$ that

(9.4-2) $$u \gtrsim v \text{ iff } \mathbb{PF}(u) \geqq \mathbb{PF}(v) .$$

When comparing two vectors of output rate histories u and v, we need not consider all the details of this preference ordering, i.e., whether future outputs are less preferred than present ones, or whether after a certain accumulated amount of a product it becomes less preferred than one with less accumulation, and the like. The intertemporal comparisons of the preference structure may be complicated indeed.

However, certain global properties will be taken for the preference function $\mathbb{PF}(u)$. The comparisons made by $\mathbb{PF}(u)$ are ordinal, i.e., $(\mathbb{PF}(u) - \mathbb{PF}(v))$ has no interpretation as excess or deficit amount of preference. Even so, as a matter of convenience we may take $\mathbb{PF}(0) = 0$, i.e., associate the real number zero with the null output vector, without loss of generality. Then, any vector $w \in (L_\infty)^m_+$ of output rate histories for which $\mathbb{PF}(w) < 0$ is less preferable than nothing at all. With some restriction of generality it will be assumed that

(9.4-3) $$\mathbb{PF}(\theta u) \geqq \mathbb{PF}(u) \quad \text{for} \quad \theta \in [1,+\infty) ,$$

which is to say that any upward scaling of a vector of output rate histories is at least as preferred as the vector being scaled. This assumption does not imply that $\mathbb{PF}(\theta u) \to +\infty$ as $\theta \to +\infty$. One need not go this far. For sufficiently large values of the scalar θ, the scaled vectors (θu) of a particular vector u may be indifferently preferred.

In place of an assumption that $\mathbb{PF}(\theta u) \to +\infty$ for $\theta \to +\infty$ for all $u \in (L_\infty)^m_+$, it is assumed instead for nonsatiation that

$$\text{(9.4-4)} \qquad \begin{array}{l} \text{For each } E \in R_+ \text{ there exists a vector} \\ u \in (L_\infty)_+^m \text{ such that } \mathbb{PF}(u) \geqq E \;. \end{array}$$

However, $\mathbb{PF}(u)$ is taken to be bounded for u bounded.

The only further assumption to be made for the preference functional $\mathbb{PF}(u)$ is that it has closed level sets, i.e.,

$$\text{(9.4-5)} \qquad A(E) := \left\{ u \in (L_\infty)_+^m : \mathbb{PF}(u) \geqq E \right\} \text{ is closed.}$$

This assumption is made primarily for mathematical convenience.

In place of the functional $\mathbb{B}\left(\frac{p}{c}, r\right)$, one may calculate

$$\mathbb{PF}^*\left(\frac{p}{c}\right) := \operatorname*{Sup}_u \left\{ \mathbb{PF}(u) : u \in \mathbb{I}\left(\frac{p}{c}\right) \right\}$$

to represent the cost-benefit relationship. Regarding $p \in (L_1)_+^n$ as a given vector of price histories for inputs, suppose $c_1 > c_2$ with $\mathbb{PF}^*\left(\frac{p}{c_1}\right) > \mathbb{PF}\left(\frac{p}{c_2}\right)$. All that one can say for such a cost-benefit relationship is that the outcome of using the larger cost c_1 is more preferred than that for the smaller outlay c_2.

Let $E \in R_+$ denote values of the preference functional $\mathbb{PF}(u)$. A preference indirect correspondence is defined by

$$\text{(9.4-6)} \qquad \begin{array}{l} E \in R_+ \to \Sigma(E) \in 2^{(L_\infty)_+^n} \text{, where} \\ \Sigma(E) = \left\{ x \in (L_\infty)_+^n : x \in \mathbb{L}(u) \;,\; \mathbb{PF}(u) \geqq E \right\} \end{array}$$

to replace the return indirect correspondence $\left(\frac{r}{R}\right) \to \Pi\left(\frac{r}{R}\right)$ (see 9.2-1,2). Alternatively $\Sigma(E)$ may be defined by

$$\text{(9.4-7)} \qquad \Sigma(E) := \bigcup_{\mathbb{P}\mathbb{F}(u) \geqq E} \mathbb{L}(u) \ .$$

An indifference class of vectors of output histories is defined by

$$\text{(9.4-8)} \qquad I(E) := \text{ISOQ}\ \Sigma(E)\ ,\ E \in R_+ \ .$$

Then $\Sigma(E)$ consists of the set of vectors of input rate histories yielding vectors of output rate histories which are at least as preferred as those of the indifference class $I(E)$.

The properties of the preference indirect production correspondence are given in the following proposition:

Proposition (9.4-1):

The properties of the correspondence $E \to \Sigma(E)$ are:

Σ.1 $\Sigma(E) = (L_\infty)^n_+$ for $E \leqq 0$, and $0 \notin \Sigma(E)$ for $E > 0$.

Σ.2 $\bigcap_{E \in R^1} \Sigma(E)$ is empty.

Σ.3 If $x \in \Sigma(E)$, $(\lambda x) \in \Sigma(E)$ for $\lambda \in [1,+\infty)$.

Σ.4 $\Sigma(E_2) \subset \Sigma(E_1)$ for $E_2 \geqq E_1$.

Σ.5 $E \to \Sigma(E)$ is Quasi-Concave in E .

Concerning Property Σ.1, clearly $\mathbb{P}\mathbb{F}(0) = 0$, and $\mathbb{L}(0) = (L_\infty)^n_+$. Hence Σ.1 holds for $E = 0$. In case $E < 0$, clearly $\mathbb{P}\mathbb{F}(0) \geqq E$ and $\mathbb{L}(0) = (L_\infty)^n_+$. Thus $\Sigma(E) = (L_\infty)^n_+$ for $E \leqq 0$. In case $E > 0$, $\mathbb{P}\mathbb{F}(u) \geqq E > 0$ implies $u \geq 0$ and $0 \notin \mathbb{L}(u)$ for $u \geq 0$. Thus $0 \notin \Sigma(E)$ for $E > 0$.

Property $\Sigma.2$ holds, because suppose $x^o \in \Sigma(E)$ for $E \in (-\infty,+\infty)$. Then $\mathbb{PF}(u)$ is unbounded on the bounded set $\mathbb{P}(x^o)$, contradicting the assumption that $\mathbb{PF}(u)$ is a finite functional.

Property $\Sigma.3$ obviously holds, since $x \in \Sigma(E)$ implies $x \in \mathbb{L}(\bar{u})$ for $\mathbb{PF}(\bar{u}) \geqq E$, while any vector $(\lambda x) \in \mathbb{L}(\bar{u})$ for $\lambda \in [1,+\infty)$ by Property $\mathbb{L}.3$.

Property $\Sigma.4$ is evident, since if $x \in \Sigma(E_2)$, $x \in \mathbb{L}(\bar{u})$ with $\mathbb{PF}(\bar{u}) \geqq E_2 \geqq E_1$, and hence $x \in \Sigma(E_1)$.

The correspondence $E \to \Sigma(E)$ is Quasi-Concave if $(\Sigma(E) \cap \Sigma(E')) \subset (\Sigma((1-\theta)E + \theta E'))$ for $\theta \in [0,1]$, $E' \neq E$. To verify $\Sigma.5$, suppose $x \in \Sigma(E) \cap \Sigma(E')$. Then $x \in \mathbb{L}(\bar{u})$ for $\mathbb{PF}(\bar{u}) \geqq E$, $\mathbb{PF}(\bar{u}) \geqq E'$, i.e., $\mathbb{PF}(\bar{u}) \geqq \text{Max}\,[E,E']$. Accordingly $\mathbb{PF}(\bar{u}) \geqq [(1-\theta)E + \theta E']$ for $\theta \in [0,1]$, and $x \in \Sigma((1-\theta)E + \theta E')$ for $\theta \in [0,1]$.

Now, with the preference indirect correspondence $E \to \Sigma(E)$, one may calculate the minimal cost functional

(9.4-9) $$\mathbb{X}(E,p) := \text{Inf}\,\{\langle p,x\rangle : x \in \Sigma(E)\}$$

to replace $\mathbb{X}\left(\frac{r}{B},p\right)$ defined by (9.3-3). This cost functional gives the minimal cost of obtaining vectors u of output rate histories which are at least as preferred as those of the indifference class $I(E)$.

The properties of $\mathbb{X}(E,p)$ in terms of $p \in (L_1)^n_+$ follow those of $\mathbb{X}\left(\frac{r}{B},p\right)$ and will not be replaced here. It is of interest to observe that

(9.4-10) $$\mathbb{X}(E',p) \geqq \mathbb{X}(E,p) \quad \text{for } E' \geqq E .$$

This property follows directly from Property $\Sigma.4$.

From an engineering viewpoint (as it sometimes appears to be practical on public projects), different values E and E' in the ordinal relationship $\mathbb{PF}(u)$ may be measured by the minimal cost of resources required to attain at least these values, assuming that the price structure p is applicable in both cases. By this treatment the cost benefit relationship

$$c \rightarrow \mathbb{PF}^*\left(\frac{p}{c}\right)$$

may be "cardinalized" by

$$c \rightarrow \mathbb{X}\left(\mathbb{PF}^*\left(\frac{p}{c}\right), p\right) . \tag{9.4-11}$$

This comparison of Maximal "Ordinal Preferences" by the resource costs of achieving them has some basis for justification provided extremes are not being compared. However, for an increased safety margin of one percent at the ninety-eight percent level, the resource cost may be very great and hardly commensurate with losses entailed by foregoing the one percent. It does not seem reasonable to assume globally that inherent worth is proportional to cost.

Still the minimal cost functional $\mathbb{X}(E,p)$ serves to provide a basis for evaluating the relative costs of two proposals ordered by the preference function $\mathbb{PF}(u)$.

REFERENCES

Färe, R. (1978:a), "Production Theory Dualities for Optimally Realized Values," in THEORY AND APPLICATIONS OF ECONOMIC INDICES, W. Eichhorn, R. Henn, O. Opitz and R. W. Shephard (editors), Physica-Verlag, Würzburg.

Färe, R. (1978:b), "Separability and Index Properties of Ray-Homothetic Dynamic Production Structures," in THEORY AND APPLICATIONS OF ECONOMIC INDICES, W. Eichhorn, R. Henn, O. Opitz and R. W. Shephard (editors), Physica-Verlag, Würzburg.

Hildenbrand, W. (1974), CORE AND EQUILIBRIA IN A LARGE ECONOMY, Princeton University Press, Princeton.

Shephard, R. W. (1953), COST AND PRODUCTION FUNCTIONS, Princeton University Press, Princeton.

Shephard, R. W. (1970:a), THEORY OF COST AND PRODUCTION FUNCTIONS, Princeton University Press, Princeton.

Shephard, R. W. (1974:a), INDIRECT PRODUCTION FUNCTIONS, Mathematical Systems in Economics, No. 10, Verlag Anton Hain, Meisenheim Am Glad.

Shephard, R. W. (1977), "Dynamic Indirect Production Functions," in MATHEMATICAL ECONOMICS AND GAME THEORY IN HONOR OF OSKAR MORGENSTERN, Lecture Notes in Economics and Mathematical Systems, Vol. 141, Springer-Verlag, Berlin.

Shephard, R. W. (1978), "A Dynamic Formulation of Index Functions for the Theory of Cost and Production," in THEORY AND APPLICATION OF ECONOMIC INDICES, W. Eichhorn, R. Henn, O. Opitz and R. W. Shephard (editors), Physica-Verlag, Würzburg.